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Derivation of angular velocity components (pqr) in body coordinates. This derivation applies to Euler angles relative to the right handed coordinate system defined by a set of axes fixed to the Earth. Yaw is the first rotation about the z axis, pitch is the second rotation about the y axis that formed following yaw, and roll is the third rotation about the x axis that was formed following pitch.

Yaw is the first rotation about the Earth's $\hat{Z}_E$ Axis. This gives, $\hat{X}_1$, $\hat{Y}_1$, and $\hat{Z}_1$ body axes in Earth coordinates:

$$\begin{matrix}\hat{X}_1\\ \hat{Y}_1\\ \hat{Z}_1\end{matrix} = |A| \cdot \begin{vmatrix}\hat{X}_E\\ \hat{Y}_E\\ \hat{Z}_E\end{vmatrix} = \begin{vmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{vmatrix} \cdot \begin{vmatrix}\hat{X}_E\\ \hat{Y}_E\\ \hat{Z}_E\end{vmatrix} \qquad 1$$

The $|A|$ matrix is the direction cosine matrix and defines the axes components after the yaw rotation. The $|B|$ and $|C|$ matrices will define the axes components after the pitch and roll rotations. The Earth axes in $\hat{X}_1$, $\hat{Y}_1$, and $\hat{Z}_1$ body coordinates are given by transposing the A matrix:

$$\begin{matrix}\hat{X}_E\\ \hat{Y}_E\\ \hat{Z}_E\end{matrix} = |A^T| \cdot \begin{vmatrix}\hat{X}_1\\ \hat{Y}_1\\ \hat{Z}_1\end{vmatrix} = \begin{vmatrix}\cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{vmatrix} \cdot \begin{vmatrix}\hat{X}_1\\ \hat{Y}_1\\ \hat{Z}_1\end{vmatrix} \qquad 2$$

This says that:

$$\begin{aligned}\hat{X}_E &= \hat{X}_1\cos\psi - \hat{Y}_1\sin\psi\\ \hat{Y}_E &= \hat{X}_1\sin\psi + \hat{Y}_1\cos\psi\\ \hat{Z}_E &= \hat{Z}_1\end{aligned} \qquad 3$$

Pitch is defined as the rotation about the new $\hat{Y}_1$ axis that was created when we yawed about the $\hat{Z}_E$ axis. We pitch about the $\hat{Y}_1$ axis to get $\hat{X}_2$, $\hat{Y}_2$, $\hat{Z}_2$ body axes.

$$\begin{matrix}\hat{X}_2\\ \hat{Y}_2\\ \hat{Z}_2\end{matrix} = |B| \cdot \begin{vmatrix}\hat{X}_1\\ \hat{Y}_1\\ \hat{Z}_1\end{vmatrix} = \begin{vmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{vmatrix} \cdot \begin{vmatrix}\hat{X}_1\\ \hat{Y}_1\\ \hat{Z}_1\end{vmatrix} \qquad 4$$

Body $\hat{X}_1$, $\hat{Y}_1$, $\hat{Z}_1$ axes in $\hat{X}_2$, $\hat{Y}_2$, $\hat{Z}_2$ coordinates are given by:

$$\begin{matrix}\hat{X}_1\\ \hat{Y}_1\\ \hat{Z}_1\end{matrix} = |B^T| \cdot \begin{vmatrix}\hat{X}_2\\ \hat{Y}_2\\ \hat{Z}_2\end{vmatrix} = \begin{vmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{vmatrix} \cdot \begin{vmatrix}\hat{X}_2\\ \hat{Y}_2\\ \hat{Z}_2\end{vmatrix} \qquad 5$$

This says that

$$\begin{aligned}\hat{X}_1 &= \hat{X}_2\cos\theta + \hat{Z}_2\sin\theta\\ \hat{Y}_1 &= \hat{Y}_2\\ \hat{Z}_1 &= -\hat{X}_2\sin\theta + \hat{Z}_2\cos\theta\end{aligned} \qquad 6$$

Roll is defined as the rotation about the new $\hat{X}_2$ axis that was created when we pitched. We roll about the $\hat{X}_2$ axis to give us the final $\hat{X}_T$, $\hat{Y}_T$, $\hat{Z}_T$ body axes in Earth coordinates.

$$\begin{matrix}\hat{X}_T\\ \hat{Y}_T\\ \hat{Z}_T\end{matrix} = |C| \cdot \begin{vmatrix}\hat{X}_2\\ \hat{Y}_2\\ \hat{Z}_2\end{vmatrix} = \begin{vmatrix}1 & 0 & 0\\ 0 & \cos\phi & \sin\phi\\ 0 & -\sin\phi & \cos\phi\end{vmatrix} \cdot \begin{vmatrix}\hat{X}_2\\ \hat{Y}_2\\ \hat{Z}_2\end{vmatrix} \quad 7$$

Body $\hat{X}_2$, $\hat{Y}_2$, $\hat{Z}_2$ axes in $\hat{X}_T$, $\hat{Y}_T$, $\hat{Z}_T$ coordinates are given by:

$$\begin{matrix}\hat{X}_2\\ \hat{Y}_2\\ \hat{Z}_2\end{matrix} = |C^T| \cdot \begin{vmatrix}\hat{X}_T\\ \hat{Y}_T\\ \hat{Z}_T\end{vmatrix} = \begin{vmatrix}1 & 0 & 0\\ 0 & \cos\phi & -\sin\phi\\ 0 & \sin\phi & \cos\phi\end{vmatrix} \cdot \begin{vmatrix}\hat{X}_T\\ \hat{Y}_T\\ \hat{Z}_T\end{vmatrix} \quad 8$$

This says that:

$$\begin{aligned}\hat{X}_2 &= \hat{X}_T\\ \hat{Y}_2 &= \hat{Y}_T \cos\phi - \hat{Z}_T \sin\phi\\ \hat{Z}_2 &= \hat{Y}_T \sin\phi + \hat{Z}_T \cos\phi\end{aligned} \quad 9$$

And

$$\begin{matrix}\hat{X}_T\\ \hat{Y}_T\\ \hat{Z}_T\end{matrix} = |C| \cdot |B| \cdot |A| \cdot \begin{vmatrix}X_E\\ Y_E\\ Z_E\end{vmatrix} \quad 10$$

Each rotation can be considered independently about the $\hat{Z}_E$, $\hat{Y}_1$, $\hat{X}_2$ axes respectively. Therefore, their rates can be summed to derive an angular velocity vector.

$$\omega = \dot{\psi}\hat{Z}_E + \dot{\theta}\hat{Y}_1 + \dot{\phi}\hat{X}_2 \quad 11$$

Where $\dot{\psi}$ is the yaw rate, $\dot{\theta}$ is the pitch rate, and $\dot{\phi}$ is the roll rate.
To create a velocity vector in body coordinates, we need to express the angular velocity components in Equation 11 in body coordinates ($\hat{X}_T$, $\hat{Y}_T$, $\hat{Z}_T$) to create the angular velocity (pqr) vector. Consider Equations 3 and 6:

$$\hat{Z}_E = \hat{Z}_1 = -\hat{X}_2 \sin\theta + \hat{Z}_2 \cos\theta \quad 12$$

Substitute $\hat{X}_2$ and $\hat{Z}_2$ from Equations 9, you get

$$\begin{aligned}\hat{Z}_E &= \hat{Z}_1 = -\hat{X}_T \sin\theta + (\hat{Y}_T \sin\phi + \hat{Z}_T \cos\phi)\cos\theta\\ \hat{Z}_E &= -\hat{X}_T \sin\theta + \hat{Y}_T \sin\phi\cos\theta + \hat{Z}_T \cos\phi\cos\theta\end{aligned} \quad 13$$

Now, we have $\hat{Z}_E$ in terms of body coordinates. Now consider the second component of Equation 11. Substitute the $\hat{Y}_1$ using Equations 6 and 9:

$$\hat{Y}_1 = \hat{Y}_2 = \hat{Y}_T \cos\phi - \hat{Z}_T \sin\phi \quad 14$$

Now, we have $\hat{Y}_1$ in terms of body coordinates. And finally consider the third component of Equation 11. Substitute the $\hat{X}_2$ with $\hat{X}_T$ from Equation 9:

$$\hat{X}_2 = \hat{X}_T \qquad 15$$

Substitute the unit vectors in equation 10 with equations 13, 14, and 15.

$$\omega = \dot{\psi}(-\hat{X}_T \sin\theta + \hat{Y}_T \sin\phi\cos\theta + \hat{Z}_T \cos\phi\cos\theta) + \dot{\theta}(\hat{Y}_T \cos\phi - \hat{Z}_T \sin\phi) + \dot{\phi}(\hat{X}_T) \qquad 16$$

Now rearrange Equation 16 to combine the $\hat{X}_T$, $\hat{Y}_T$, $\hat{Z}_T$ components so

$$\omega = \hat{X}_T(\dot{\phi} - \dot{\psi}\sin\theta) + \hat{Y}_T(\dot{\theta}\cos\phi + \dot{\psi}\sin\phi\cos\theta) + \hat{Z}_T(-\dot{\theta}\sin\phi + \dot{\psi}\cos\phi\cos\theta) \qquad 17$$

Angular velocity is a vector with components that are functions of pitch, roll, yaw rate, pitch rate, and roll rate. The resulting components agree with the ω_x, ω_y, and ω_z values given in the IEEE std. P1278.1/d8.