# Quaternions

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## Definition

$$\boldsymbol{q} \;=\; q_w \;+\; \hat{\boldsymbol{i}}\,q_x \;+\; \hat{\boldsymbol{j}}\,q_y \;+\; \hat{\boldsymbol{k}}\,q_z \tag{1}$$

## Product of Two Quaternions

$$\begin{aligned}
\boldsymbol{q}\,\boldsymbol{p} &= (q_w + \hat{\boldsymbol{i}}q_x + \hat{\boldsymbol{j}}q_y + \hat{\boldsymbol{k}}q_z)(p_w + \hat{\boldsymbol{i}}\,p_x + \hat{\boldsymbol{j}}\,p_y + \hat{\boldsymbol{k}}\,p_z) \\
&= q_w p_w + \hat{\boldsymbol{i}}q_w p_x + \hat{\boldsymbol{j}}q_w p_y + \hat{\boldsymbol{k}}q_w p_z + \hat{\boldsymbol{i}}q_x p_w + \hat{\boldsymbol{i}}*\hat{\boldsymbol{i}}\,q_x p_x + \hat{\boldsymbol{i}}*\hat{\boldsymbol{j}}q_x p_y + \hat{\boldsymbol{i}}*\hat{\boldsymbol{k}}\,q_x p_z \\
&\quad + \hat{\boldsymbol{j}}\,q_y p_w + \hat{\boldsymbol{j}}*\hat{\boldsymbol{i}}\,q_y p_x + \hat{\boldsymbol{j}}*\hat{\boldsymbol{j}}\,q_y p_y + \hat{\boldsymbol{j}}*\hat{\boldsymbol{k}}\,q_y p_z \\
&\quad + \hat{\boldsymbol{k}}\,q_z p_w + \hat{\boldsymbol{k}}*\hat{\boldsymbol{i}}\,q_z p_x + \hat{\boldsymbol{k}}*\hat{\boldsymbol{j}}\,q_z p_y + \hat{\boldsymbol{k}}*\hat{\boldsymbol{k}}\,q_z p_z \\[1em]
&= q_w p_w + \hat{\boldsymbol{i}}q_w p_x + \hat{\boldsymbol{j}}q_w p_y + \hat{\boldsymbol{k}}q_w p_z + \hat{\boldsymbol{i}}q_x p_w - q_x p_x + \hat{\boldsymbol{k}}q_x p_y - \hat{\boldsymbol{j}}\,q_x p_z \\
&\quad + \hat{\boldsymbol{j}}\,q_y p_w - \hat{\boldsymbol{k}}q_y p_x - q_y p_y + \hat{\boldsymbol{i}}\,q_y p_z + \hat{\boldsymbol{k}}\,q_z p_w + \hat{\boldsymbol{j}}\,q_z p_x - \hat{\boldsymbol{i}}\,q_z p_y - q_z p_z
\end{aligned}$$

So,

$$\begin{aligned}
\boldsymbol{q}\,\boldsymbol{p} &= (q_w p_w - q_x p_x - q_y p_y - q_z p_z) + \hat{\boldsymbol{i}}(q_w p_x + q_x p_w + q_y p_z - q_z p_y) \\
&\quad + \hat{\boldsymbol{j}}(q_w p_y - q_x p_z + q_y p_w + q_z p_x) + \hat{\boldsymbol{k}}(q_w p_z + q_x p_y - q_y p_x + q_z p_w)
\end{aligned} \tag{2}$$

## Product of a Quaternion and a Vector

$$\begin{aligned}
\vec{d}\,\boldsymbol{q} &= (-v_x q_x - v_y q_y - v_z q_z) + \hat{\boldsymbol{i}}(v_x q_w + v_y q_z - v_z q_y) \\
&\quad + \hat{\boldsymbol{j}}(-v_x q_z + v_y q_w + v_z q_x) + \hat{\boldsymbol{k}}(v_x q_y - v_y q_x + v_z q_w)
\end{aligned} \tag{3}$$

$$\boldsymbol{q}\vec{d} = (-q_x v_x - q_y v_y - q_z v_z) + \hat{\boldsymbol{i}}(q_w v_x + q_y v_z - q_z v_y) \\ + \hat{\boldsymbol{j}}(q_w v_y - q_x v_z + q_z v_x) + \hat{\boldsymbol{k}}(q_w v_z + q_x v_y - q_y v_x) \tag{4}$$

$$\boldsymbol{q}^\dagger\vec{d} = (-q_x v_x + q_y v_y + q_z v_z) + \hat{\boldsymbol{i}}(q_w v_x - q_y v_z + q_z v_y) \\ + \hat{\boldsymbol{j}}(q_w v_y + q_x v_z - q_z v_x) + \hat{\boldsymbol{k}}(q_w v_z - q_x v_y + q_y v_x) \tag{5}$$

## Conjugate

$$\boldsymbol{q}^\dagger = q_w - \hat{\boldsymbol{i}}q_x - \hat{\boldsymbol{j}}q_y - \hat{\boldsymbol{k}}q_z \tag{6}$$

$$\begin{aligned}\boldsymbol{q}\boldsymbol{q}^\dagger &= (q_w q_w + q_x q_x + q_y q_y + q_z q_z) + \hat{\boldsymbol{i}}(-q_w q_x + q_x q_w - q_y q_z + q_z q_y) \\ &\quad + \hat{\boldsymbol{j}}(-q_w q_y + q_x q_z + q_y q_w - q_z q_x) + \hat{\boldsymbol{k}}(-q_w q_z - q_x q_y + q_y q_x + q_z q_w) \\ &= q_w^2 + q_x^2 + q_y^2 + q_z^2\end{aligned} \tag{7}$$

For a vector (i.e., a quaternion with no scalar part),

$$\vec{v}^\dagger = -\vec{v} \tag{8}$$

$$\begin{aligned}\vec{v}_1 * \vec{v}_2 &= (-v_{1x}v_{2x} - v_{1y}v_{2y} - v_{1z}v_{2z}) + \hat{\boldsymbol{i}}(v_{1y}v_{2z} - v_{1z}v_{2y}) + \hat{\boldsymbol{j}}(-v_{1x}v_{2z} + v_{1z}v_{2x}) + \hat{\boldsymbol{k}}(v_{1x}v_{2y} - v_{1y}v_{2x}) \\ \vec{v}_2 * \vec{v}_1 &= (-v_{2x}v_{1x} - v_{2y}v_{1y} - v_{2z}v_{1z}) + \hat{\boldsymbol{i}}(v_{2y}v_{1z} - v_{2z}v_{1y}) + \hat{\boldsymbol{j}}(-v_{2x}v_{1z} + v_{2z}v_{1x}) + \hat{\boldsymbol{k}}(v_{2x}v_{1y} - v_{2y}v_{1x})\end{aligned} \tag{9}$$

$$\vec{v}_2 * \vec{v}_1 = -\vec{v}_1 * \vec{v}_2 \tag{10}$$

## Useful Formulas

$$\boldsymbol{p}\boldsymbol{q} = (p_w q_w - p_x q_x - p_y q_y - p_z q_z) + \hat{\boldsymbol{i}}(p_w q_x + p_x q_w + p_y q_z - p_z q_y) \\ + \hat{\boldsymbol{j}}(p_w q_y - p_x q_z + p_y q_w + p_z q_x) + \hat{\boldsymbol{k}}(p_w q_z + p_x q_y - p_y q_x + p_z q_w) \tag{11}$$

$$\begin{aligned}(\boldsymbol{p}\boldsymbol{q})^\dagger &= (p_w q_w - p_x q_x - p_y q_y - p_z q_z) + \hat{\boldsymbol{i}}(-p_w q_x - p_x q_w - p_y q_z + p_z q_y) \\ &\quad + \hat{\boldsymbol{j}}(-p_w q_y + p_x q_z - p_y q_w - p_z q_x) + \hat{\boldsymbol{k}}(-p_w q_z - p_x q_y + p_y q_x - p_z q_w) \\ &= (q_w p_w - q_x p_x - q_y p_y - q_z p_z) + \hat{\boldsymbol{i}}(-q_x p_w - q_w p_x - q_z p_y + q_y p_z) \\ &\quad + \hat{\boldsymbol{j}}(-q_y p_w + q_z p_x - q_w p_y - q_x p_z) + \hat{\boldsymbol{k}}(-q_z p_w - q_y p_x + q_x p_y - q_w p_z) \\ &= (q_w p_w - q_x p_x - q_y p_y - q_z p_z) + \hat{\boldsymbol{i}}(-q_w p_x - q_x p_w + q_y p_z - q_z p_y) \\ &\quad + \hat{\boldsymbol{j}}(-q_w p_y - q_x p_z - q_y p_w + q_z p_x) + \hat{\boldsymbol{k}}(-q_w p_z + q_x p_y - q_y p_x - q_z p_w)\end{aligned} \tag{12}$$

$$\boldsymbol{q}^\dagger\boldsymbol{p}^\dagger = (q_w p_w - q_x p_x - q_y p_y - q_z p_z) + \hat{\boldsymbol{i}}(-q_w p_x - q_x p_w + q_y p_z - q_z p_y) \\ + \hat{\boldsymbol{j}}(-q_w p_y - q_x p_z - q_y p_w + q_z p_x) + \hat{\boldsymbol{k}}(-q_w p_z + q_x p_y - q_y p_x - q_z p_w) \tag{13}$$

So,

$$(\boldsymbol{p}\,\boldsymbol{q})^{\dagger} \;=\; \boldsymbol{q}^{\dagger}\,\boldsymbol{p}^{\dagger} \tag{14}$$

## Quaternions vs Matrix Multiplication

We can show that a quaternion transformation is equivalent to a matrix multiplication.

$$\begin{aligned}
\vec{\boldsymbol{v}}\,' \;=\; \boldsymbol{q}\,\vec{\boldsymbol{v}}\,\boldsymbol{q}^{\dagger} \;&=\; (q_w+\hat{\boldsymbol{i}}\,q_x+\hat{\boldsymbol{j}}\,q_y+\hat{\boldsymbol{k}}\,q_z)(\hat{\boldsymbol{i}}\,v_x+\hat{\boldsymbol{j}}\,v_y+\hat{\boldsymbol{k}}\,v_z)(q_w-\hat{\boldsymbol{i}}\,q_x-\hat{\boldsymbol{j}}\,q_y-\hat{\boldsymbol{k}}\,q_z)\\
&=\; (\hat{\boldsymbol{i}}\,q_w v_x+\hat{\boldsymbol{j}}\,q_w v_y+\hat{\boldsymbol{k}}\,q_w v_z-q_x v_x+\hat{\boldsymbol{k}}\,q_x v_y-\hat{\boldsymbol{j}}\,q_x v_z-\hat{\boldsymbol{k}}\,q_y v_x-q_y v_y+\hat{\boldsymbol{i}}\,q_y v_z\\
&\quad +\hat{\boldsymbol{j}}\,q_z v_x-\hat{\boldsymbol{i}}\,q_z v_y-q_z v_z)(q_w-\hat{\boldsymbol{i}}\,q_x-\hat{\boldsymbol{j}}\,q_y-\hat{\boldsymbol{k}}\,q_z)\\
&=\; [\;(-q_x v_x-q_y v_y-q_z v_z)+\hat{\boldsymbol{i}}(q_w v_x+q_y v_z-q_z v_y)+\hat{\boldsymbol{j}}(q_w v_y-q_x v_z+q_z v_x)\\
&\quad +\hat{\boldsymbol{k}}(q_w v_z+q_x v_y-q_y v_x)\,](q_w-\hat{\boldsymbol{i}}\,q_x-\hat{\boldsymbol{j}}\,q_y-\hat{\boldsymbol{k}}\,q_z)\\
&=\; (-q_x v_x-q_y v_y-q_z v_z)(q_w-\hat{\boldsymbol{i}}\,q_x-\hat{\boldsymbol{j}}\,q_y-\hat{\boldsymbol{k}}\,q_z)\\
&\quad +\hat{\boldsymbol{i}}(q_w v_x+q_y v_z-q_z v_y)(q_w-\hat{\boldsymbol{i}}\,q_x-\hat{\boldsymbol{j}}\,q_y-\hat{\boldsymbol{k}}\,q_z)\\
&\quad +\hat{\boldsymbol{j}}(q_w v_y-q_x v_z+q_z v_x)(q_w-\hat{\boldsymbol{i}}\,q_x-\hat{\boldsymbol{j}}\,q_y-\hat{\boldsymbol{k}}\,q_z)\\
&\quad +\hat{\boldsymbol{k}}(q_w v_z+q_x v_y-q_y v_x)(q_w-\hat{\boldsymbol{i}}\,q_x-\hat{\boldsymbol{j}}\,q_y-\hat{\boldsymbol{k}}\,q_z)\\
&=\; (-q_w q_x v_x-q_w q_y v_y-q_w q_z v_z)+\hat{\boldsymbol{i}}(q_x^2 v_x+q_x q_y v_y+q_x q_z v_z)\\
&\quad +\hat{\boldsymbol{j}}(q_x q_y v_x+q_y^2 v_y+q_y q_z v_z)+\hat{\boldsymbol{k}}(q_x q_z v_x+q_y q_z v_y+q_z^2 v_z)\\
&\quad +\hat{\boldsymbol{i}}(q_w^2 v_x+q_w q_y v_z-q_w q_z v_y)+(q_w q_x v_x+q_x q_y v_z-q_x q_z v_y)\\
&\quad -\hat{\boldsymbol{k}}(q_w q_y v_x+q_y^2 v_z-q_y q_z v_y)+\hat{\boldsymbol{j}}(q_w q_z v_x+q_y q_z v_z-q_z^2 v_y)\\
&\quad +\hat{\boldsymbol{j}}(q_w^2 v_y-q_w q_x v_z+q_w q_z v_x)+\hat{\boldsymbol{k}}(q_w q_x v_y-q_x^2 v_z+q_x q_z v_x)\\
&\quad +(q_w q_y v_y-q_x q_y v_z+q_y q_z v_x)-\hat{\boldsymbol{i}}(q_w q_z v_y-q_x q_z v_z+q_z^2 v_x)\\
&\quad +\hat{\boldsymbol{k}}(q_w^2 v_z+q_w q_x v_y-q_w q_y v_x)-\hat{\boldsymbol{j}}(q_w q_x v_z+q_x^2 v_y-q_x q_y v_x)\\
&\quad +\hat{\boldsymbol{i}}(q_w q_y v_z+q_x q_y v_y-q_y^2 v_x)+(q_w q_z v_z+q_x q_z v_y-q_y q_z v_x)\\
&=\; \hat{\boldsymbol{i}}(+q_w^2 v_x+q_x^2 v_x-q_y^2 v_x-q_z^2 v_x+2q_x q_y v_y-2q_w q_z v_y+2q_x q_z v_z+2q_w q_y v_z)\\
&\quad +\hat{\boldsymbol{j}}(2q_w q_z v_x 2q_x q_y v_x+q_w^2 v_y-q_x^2 v_y+q_y^2 v_y-q_z^2 v_y+2q_y q_z v_z-2q_w q_x v_z)\\
&\quad +\hat{\boldsymbol{k}}(2\mathrm{q}_x q_z v_x-2\mathrm{q}_w q_y v_x+2q_y q_z v_y+2q_w q_x v_y+q_w^2 v_z-q_x^2 v_z-q_y^2 v_z+q_z^2 v_z)
\end{aligned}$$

Because we're working with normalised quaternions,

$$\begin{aligned}
\vec{\boldsymbol{v}}\,' \;=\; &\hat{\boldsymbol{i}}[(1-2q_y^2-2q_z^2)v_x+2(q_x q_y-q_w q_z)v_y+2(q_x q_z+q_w q_y)v_z]\\
&+\hat{\boldsymbol{j}}[2(q_w q_z+q_x q_y)v_x(1-2q_x^2-2\mathrm{q}_z^2)v_y+2(q_y q_z-q_w q_x)v_z]\\
&+\hat{\boldsymbol{k}}[2(q_x q_z-q_w q_y)v_x+2(q_y q_z+q_w q_x)v_y+(1-2q_x^2-2q_y^2)v_z]
\end{aligned}$$

Which is the same as multiplying by the matrix:

$$\vec{\boldsymbol{v}}\,' \;=\; \begin{pmatrix} 1-2q_y^2-2q_z^2 & 2(q_x q_y-q_w q_z) & 2(q_x q_z+q_w q_y)\\ 2(q_w q_z+q_x q_y) & 1-2q_x^2-2\mathrm{q}_z^2 & 2(q_y q_z-q_w q_x)\\ 2(q_x q_z-q_w q_y) & 2(q_y q_z+q_w q_x) & 1-2q_x^2-2q_y^2 \end{pmatrix}\begin{pmatrix} v_x\\ v_y\\ v_z\end{pmatrix}$$

## Combining Rotations

We can easily derive the formula for a quaternion that represents a general rotation. Let $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$ to represent three mutually perpendicular unit vectors, where we rotate first by around $\hat{e}_1$, then $\hat{e}_2$, and finally $\hat{e}_3$. The rotation is:

$$R_3 R_2 R_1 = (\cos\frac{\theta_3}{2} + \hat{\boldsymbol{e}}_3 \sin\frac{\theta_3}{2})(\cos\frac{\theta_2}{2} + \hat{\boldsymbol{e}}_2 \sin\frac{\theta_2}{2})(\cos\frac{\theta_1}{2} + \hat{\boldsymbol{e}}_1 \sin\frac{\theta_1}{2})$$

$$R_3 R_2 R_1 = (\cos\frac{\theta_3}{2}\cos\frac{\theta_2}{2} + \hat{\boldsymbol{e}}_2 \cos\frac{\theta_3}{2}\sin\frac{\theta_2}{2} + \hat{\boldsymbol{e}}_3 \sin\frac{\theta_3}{2}\cos\frac{\theta_2}{2} + \hat{\boldsymbol{e}}_3 * \hat{\boldsymbol{e}}_2 \sin\frac{\theta_3}{2}\sin\frac{\theta_2}{2})(\cos\frac{\theta_1}{2} + \hat{\boldsymbol{e}}_1 \sin\frac{\theta_1}{2})$$

$$\begin{aligned} R_3 R_2 R_1 = {} & \cos\frac{\theta_3}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_1}{2} + \hat{\boldsymbol{e}}_1 \cos\frac{\theta_3}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2} + \hat{\boldsymbol{e}}_2 \cos\frac{\theta_3}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2} + \hat{\boldsymbol{e}}_2 * \hat{\boldsymbol{e}}_1 \cos\frac{\theta_3}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2} \\ & + \hat{\boldsymbol{e}}_3 \sin\frac{\theta_3}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_1}{2} + \hat{\boldsymbol{e}}_3 * \hat{\boldsymbol{e}}_1 \sin\frac{\theta_3}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2} + \hat{\boldsymbol{e}}_3 * \hat{\boldsymbol{e}}_2 \sin\frac{\theta_3}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2} + \hat{\boldsymbol{e}}_3 * \hat{\boldsymbol{e}}_2 * \hat{\boldsymbol{e}}_1 \sin\frac{\theta_3}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2} \end{aligned} \qquad (15)$$

It will be convenient to define the variable $e$

$$\hat{\boldsymbol{e}}_3 * \hat{\boldsymbol{e}}_2 \equiv e \hat{\boldsymbol{e}}_1 \qquad (16)$$

where e = ±1 depending on the order of rotation. We also know

$$\hat{\boldsymbol{e}}_3 * \hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{e}}_2 * \hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{e}}_1 * \hat{\boldsymbol{e}}_1 = -1 \qquad (17)$$

Then

$$\begin{aligned} & \hat{\boldsymbol{e}}_3 * \hat{\boldsymbol{e}}_2 * \hat{\boldsymbol{e}}_1 = e \hat{\boldsymbol{e}}_1 * \hat{\boldsymbol{e}}_1 = -e \\ & \hat{\boldsymbol{e}}_2 * \hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{e}}_2 * (\frac{1}{e}\hat{\boldsymbol{e}}_3 * \hat{\boldsymbol{e}}_2) = -\frac{1}{e}\hat{\boldsymbol{e}}_2 * \hat{\boldsymbol{e}}_2 * \hat{\boldsymbol{e}}_3 = \frac{1}{e}\hat{\boldsymbol{e}}_3 = \frac{e}{e^2}\hat{\boldsymbol{e}}_3 = e\hat{\boldsymbol{e}}_3 \\ & \hat{\boldsymbol{e}}_3 * \hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{e}}_3 * (\frac{1}{e}\hat{\boldsymbol{e}}_3 * \hat{\boldsymbol{e}}_2) = \frac{1}{e}\hat{\boldsymbol{e}}_3 * \hat{\boldsymbol{e}}_3 * \hat{\boldsymbol{e}}_2 = -\frac{1}{e}\hat{\boldsymbol{e}}_2 = -\frac{e}{e^2}\hat{\boldsymbol{e}}_2 = -e\hat{\boldsymbol{e}}_2 \end{aligned} \qquad (18)$$

My logic in that step may be sloppy because I've taken cross products of scalars and vectors, and I'm not sure that's meaningful. However, it works in this case. See the end of this document for a more detailed proof.

Substituting into (15),

$$\begin{aligned} R_3 R_2 R_1 = {} & \cos\frac{\theta_3}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_1}{2} + \hat{\boldsymbol{e}}_1 \cos\frac{\theta_3}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2} + \hat{\boldsymbol{e}}_2 \cos\frac{\theta_3}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2} + e\hat{\boldsymbol{e}}_3 \cos\frac{\theta_3}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2} \\ & + \hat{\boldsymbol{e}}_3 \sin\frac{\theta_3}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_1}{2} - e\hat{\boldsymbol{e}}_2 \sin\frac{\theta_3}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2} + e\hat{\boldsymbol{e}}_1 \sin\frac{\theta_3}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2} - e \sin\frac{\theta_3}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2} \end{aligned}$$

And rearranging terms gives

$$\begin{aligned} R_3 R_2 R_1 = {} & (\cos\frac{\theta_3}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_1}{2} - e\sin\frac{\theta_3}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2}) \\ & + \hat{\boldsymbol{e}}_1 (\cos\frac{\theta_3}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2} + e\sin\frac{\theta_3}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2}) \\ & + \hat{\boldsymbol{e}}_2 (\cos\frac{\theta_3}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2} - e\sin\frac{\theta_3}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2}) \\ & + \hat{\boldsymbol{e}}_3 (\sin\frac{\theta_3}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_1}{2} + e\cos\frac{\theta_3}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2}) \end{aligned} \qquad (19)$$

In this derivation we assumed that the rotations around three mutually perpendicular axes. So while this equation can be adapted to rotations in any order (by setting the value of e appropriately), it is not valid if two of the rotation axes are the same. (E.g., if we rotate around the y-axis, then around the x-axis, then around the y-axis again, the equation is not valid... or is it???? Need to review this.)

## General Derivation for any Rotation Order

From (19), the components of the resulting quaternion are given by

$$
\begin{aligned}
p_0 &= \cos\frac{\theta_3}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_1}{2} - e\sin\frac{\theta_3}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2} \\
p_1 &= \cos\frac{\theta_3}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2} + e\sin\frac{\theta_3}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2} \\
p_2 &= \cos\frac{\theta_3}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2} - e\sin\frac{\theta_3}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2} \\
p_3 &= \sin\frac{\theta_3}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_1}{2} + e\cos\frac{\theta_3}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2}
\end{aligned} \tag{20}
$$

**Note:** Although $p_0$ is the scalar component of the quaternion, and $p_1$, $p_2$, $p_3$ are the remaining components, the subscripts indicate the order of rotation. So for example, if the rotation is around the y-axis, then around the x-axis, then around the z-axis, the quaternion representing the combined rotation is ( $p_0$, $p_2$, $p_1$, $p_3$), not ($p_0$, $p_1$, $p_2$, $p_3$)!

To make the derivations easier, we'll use the notation:

$$
\begin{aligned}
c_1 = \cos\frac{\theta_1}{2} \qquad & s_1 = \sin\frac{\theta_1}{2} \\
\textit{Similarly for } c_{2,} s_{2,} c_3 & \textit{ and } s_3
\end{aligned} \tag{21}
$$

The half-angle formulas can be written:

$$
s_1 c_1 = \frac{1}{2}\sin\theta_1 \qquad c_1^2 - s_1^2 = \cos\theta_1 \tag{22}
$$

So (20) can be written

$$
\begin{aligned}
p_0 &= c_3 c_2 c_1 - e s_3 s_2 s_1 \\
p_1 &= c_3 c_2 s_1 + e s_3 s_2 c_1 \\
p_2 &= c_3 s_2 c_1 - e s_3 c_2 s_1 \\
p_3 &= s_3 c_2 c_1 + e c_3 s_2 s_1
\end{aligned} \tag{23}
$$

The squares of the components will be useful in our derivation. Note that because the variable e only takes on the values ±1, we know $e^2 = 1$.

$$
\begin{aligned}
p_0^2 &= c_3^2 c_2^2 c_1^2 - 2e c_3 s_3 c_2 s_2 c_1 s_1 + s_3^2 s_2^2 s_1^2 \\
p_1^2 &= c_3^2 c_2^2 s_1^2 + 2e c_3 s_3 c_2 s_2 c_1 s_1 + s_3^2 s_2^2 c_1^2 \\
p_2^2 &= c_3^2 s_2^2 c_1^2 - 2e c_3 s_3 s_2 c_2 c_1 s_1 + s_3^2 c_2^2 s_1^2 \\
p_3^2 &= s_3^2 c_2^2 c_1^2 + 2e c_3 s_3 c_2 s_2 c_1 s_1 + c_3^2 s_2^2 s_1^2
\end{aligned} \tag{24}
$$

One more useful expression is

$$
\begin{aligned}
\cos\theta_a \cos\theta_b &= (c_a^2 - s_a^2)(c_b^2 - s_b^2) \\
&= c_a^2 c_b^2 - c_a^2 s_b^2 - s_a^2 c_b^2 + s_a^2 s_b^2 \\
&= (1 - s_a^2) c_b^2 - c_a^2 s_b^2 - s_a^2 c_b^2 + (1 - c_a^2) s_b^2 \\
&= c_b^2 - s_a^2 c_b^2 - c_a^2 s_b^2 - s_a^2 c_b^2 + s_b^2 - c_a^2 s_b^2 \\
&= (c_b^2 + s_b^2) - 2 s_a^2 c_b^2 - 2 c_a^2 s_b^2 \\
&= 1 - 2(s_a^2 c_b^2 + c_a^2 s_b^2)
\end{aligned} \tag{25}
$$

$$
s_a^2 c_b^2 + c_a^2 s_b^2 = \frac{1}{2}(1 - \cos\theta_a \cos\theta_b)
$$

## *Calculating $\theta_3$*

From (23),

$$
\begin{aligned}
p_0 p_3 &= (c_3 c_2 c_1 - e s_3 s_2 s_1)(s_3 c_2 c_1 + e c_3 s_2 s_1) \\
&= c_3 s_3 c_2^2 c_1^2 + e c_3^2 c_2 s_2 c_1 s_1 - e s_3^2 c_2 s_2 c_1 s_1 - c_3 s_3 s_2^2 s_1^2
\end{aligned}
$$

Also from (23),

$$
\begin{aligned}
p_1 p_2 &= (c_3 c_2 s_1 + e s_3 s_2 c_1)(c_3 s_2 c_1 - e s_3 c_2 s_1) \\
&= c_3^2 c_2 s_2 c_1 s_1 - e c_3 s_3 c_2^2 s_1^2 + e c_3 s_3 s_2^2 c_1^2 - s_3^2 c_2 s_2 c_1 s_1
\end{aligned}
$$

Combining the two terms,

$$
\begin{aligned}
p_0 p_3 - e p_1 p_2 &= c_3 s_3 c_2^2 c_1^2 + \cancel{e c_3^2 c_2 s_2 c_1 s_1} - \cancel{e s_3^2 c_2 s_2 c_1 s_1} - c_3 s_3 s_2^2 s_1^2 \\
&\quad - \cancel{e c_3^2 c_2 s_2 c_1 s_1} + c_3 s_3 c_2^2 s_1^2 - c_3 s_3 s_2^2 c_1^2 + \cancel{e s_3^2 c_2 s_2 c_1 s_1} \\
&= c_3 s_3 c_2^2 c_1^2 - c_3 s_3 s_2^2 s_1^2 + c_3 s_3 c_2^2 s_1^2 - c_3 s_3 s_2^2 c_1^2 \\
&= c_3 s_3 c_2^2 c_1^2 + c_3 s_3 c_2^2 s_1^2 - c_3 s_3 s_2^2 s_1^2 - c_3 s_3 s_2^2 c_1^2 \\
&= c_3 s_3 c_2^2 (c_1^2 + s_1^2) - c_3 s_3 s_2^2 (s_1^2 + c_1^2) \\
&= c_3 s_3 c_2^2 - c_3 s_3 s_2^2 \\
&= c_3 s_3 (c_2^2 - s_2^2)
\end{aligned}
$$

Using (22)

$$
p_0 p_3 - e p_1 p_2 = \frac{1}{2} \sin\theta_3 \cos\theta_2
$$

$$
\sin\theta_3 \cos\theta_2 = 2(p_0 p_3 - e p_1 p_2) \tag{26}
$$

From (24),

$$
\begin{aligned}
p_2^2 + p_3^2 &= c_3^2 s_2^2 c_1^2 - 2 e c_3 s_3 s_2 c_2 c_1 s_1 + s_3^2 c_2^2 s_1^2 + s_3^2 c_2^2 c_1^2 + 2 e c_3 s_3 c_2 s_2 c_1 s_1 + c_3^2 s_2^2 s_1^2 \\
&= c_3^2 s_2^2 c_1^2 + s_3^2 c_2^2 s_1^2 + s_3^2 c_2^2 c_1^2 + c_3^2 s_2^2 s_1^2 \\
&= c_3^2 s_2^2 c_1^2 + c_3^2 s_2^2 s_1^2 + s_3^2 c_2^2 c_1^2 + s_3^2 c_2^2 s_1^2 \\
&= c_3^2 s_2^2 (c_1^2 + s_1^2) + s_3^2 c_2^2 (c_1^2 + s_1^2) \\
&= c_3^2 s_2^2 + s_3^2 c_2^2
\end{aligned}
$$

Using (25),

$$
\begin{aligned}
p_2^2 + p_3^2 &= \frac{1}{2}(1 - \cos\theta_3 \cos\theta_2) \\
2(p_2^2 + p_3^2) &= 1 - \cos\theta_3 \cos\theta_2
\end{aligned}
$$

$$
\cos\theta_3 \cos\theta_2 = 1 - 2(p_2^2 + p_3^2) \tag{27}
$$

Combining with (26),

$$
\tan\theta_3 = \frac{2(p_0 p_3 - e p_1 p_2)}{1 - 2(p_2^2 + p_3^2)} \tag{28}
$$

This should be calculated using a two-argument arctangent function. There is a singularity where both the numerator and the denominator are zero. Examining (26) and (27), we can see that this can only occur if $\cos\theta_2$ is zero, which means $\theta_2 = \pm 90°$. We'll deal with the singularities later.

## *Calculating $\theta_2$*

From (23),

$$
\begin{aligned}
p_0 p_2 &= (c_3 c_2 c_1 - e s_3 s_2 s_1)(c_3 s_2 c_1 - e s_3 c_2 s_1) \\
&= c_3^2 c_2 s_2 c_1^2 - e c_3 s_3 c_2^2 c_1 s_1 - e c_3 s_3 s_2^2 c_1 s_1 + s_3^2 c_2 s_2 s_1^2
\end{aligned}
$$

Also from (23),

$$\begin{aligned} p_1 p_3 &= (c_3 c_2 s_1 + e s_3 s_2 c_1)(s_3 c_2 c_1 + e c_3 s_2 s_1) \\ &= c_3 s_3 c_2^2 c_1 s_1 + e c_3^2 c_2 s_2 s_1^2 + e s_3^2 c_2 s_2 c_1^2 + c_3 s_3 s_2^2 c_1 s_1 \end{aligned}$$

Combining the two terms,

$$\begin{aligned} p_0 p_2 + e p_1 p_3 &= c_3^2 c_2 s_2 c_1^2 - \cancel{e e c_3 s_3 c_2^2 c_1 s_1} - \cancel{e e c_3 s_3 s_2^2 c_1 s_1} + s_3^2 c_2 s_2 s_1^2 \\ &\quad + \cancel{e e c_3 s_3 c_2^2 c_1 s_1} + c_3^2 c_2 s_2 s_1^2 + s_3^2 c_2 s_2 c_1^2 + \cancel{e e c_3 s_3 s_2^2 c_1 s_1} \\ &= c_3^2 c_2 s_2 c_1^2 + s_3^2 c_2 s_2 s_1^2 + c_3^2 c_2 s_2 s_1^2 + s_3^2 c_2 s_2 c_1^2 \\ &= c_3^2 c_2 s_2 (c_1^2 + s_1^2) + s_3^2 c_2 s_2 (s_1^2 + c_1^2) \\ &= c_3^2 c_2 s_2 + s_3^2 c_2 s_2 \\ &= c_2 s_2 (c_3^2 + s_3^2) \\ &= c_2 s_2 \end{aligned}$$

Using (22)

$$p_0 p_2 + e p_1 p_3 = \frac{1}{2} \sin \theta_2$$

$$\sin \theta_2 = 2(p_0 p_2 + e p_1 p_3) \tag{29}$$

## *Calculating θ₁*

From (24),

$$\begin{aligned} p_1^2 + p_2^2 &= c_3^2 c_2^2 s_1^2 + 2 e c_3 s_3 c_2 s_2 s_1 c_1 + s_3^2 s_2^2 c_1^2 + c_3^2 s_2^2 c_1^2 - 2 e c_3 s_3 s_2 c_2 c_1 s_1 + s_3^2 c_2^2 s_1^2 \\ &= c_3^2 c_2^2 s_1^2 + c_3^2 s_2^2 c_1^2 + s_3^2 s_2^2 c_1^2 + s_3^2 c_2^2 s_1^2 \\ &= c_3^2 (c_2^2 s_1^2 + s_2^2 c_1^2) + s_3^2 (s_2^2 c_1^2 + c_2^2 s_1^2) \\ &= (c_3^2 + s_3^2)(c_2^2 s_1^2 + s_2^2 c_1^2) \\ &= c_2^2 s_1^2 + s_2^2 c_1^2 \end{aligned}$$

Using (25),

$$p_1^2 + p_2^2 = \frac{1}{2}(1 - \cos\theta_2 \cos\theta_1)$$

$$2(p_1^2 + p_2^2) = 1 - \cos\theta_2 \cos\theta_1$$

$$\cos\theta_2 \cos\theta_1 = 1 - 2(p_1^2 + p_2^2) \tag{30}$$

From (23),

$$\begin{aligned} p_0 p_1 &= (c_3 c_2 c_1 - e s_3 s_2 s_1)(c_3 c_2 s_1 + e s_3 s_2 c_1) \\ &= c_3^2 c_2^2 c_1 s_1 + e c_3 s_3 c_2 s_2 c_1^2 - e c_3 s_3 c_2 s_2 s_1^2 - s_3^2 s_2^2 c_1 s_1 \end{aligned}$$

Also from (23),

$$\begin{aligned} p_2 p_3 &= (c_3 s_2 c_1 - e s_3 c_2 s_1)(s_3 c_2 c_1 + e c_3 s_2 s_1) \\ &= c_3 s_3 c_2 s_2 c_1^2 + e c_3^2 s_2^2 c_1 s_1 - e s_3^2 c_2^2 c_1 s_1 - c_3 s_3 c_2 s_2 s_1^2 \end{aligned}$$

Combining the two terms,

$$\begin{aligned} p_0 p_1 - e p_2 p_3 &= c_3^2 c_2^2 c_1 s_1 + \cancel{e c_3 s_3 c_2 s_2 c_1^2} - \cancel{e c_3 s_3 c_2 s_2 s_1^2} - s_3^2 s_2^2 c_1 s_1 \\ &\quad - \cancel{e c_3 s_3 c_2 s_2 c_1^2} - c_3^2 s_2^2 c_1 s_1 + s_3^2 c_2^2 c_1 s_1 + \cancel{e c_3 s_3 c_2 s_2 s_1^2} \\ &= c_3^2 c_2^2 c_1 s_1 - s_3^2 s_2^2 c_1 s_1 - c_3^2 s_2^2 c_1 s_1 + s_3^2 c_2^2 c_1 s_1 \\ &= c_3^2 c_2^2 c_1 s_1 + s_3^2 c_2^2 c_1 s_1 - s_3^2 s_2^2 c_1 s_1 - c_3^2 s_2^2 c_1 s_1 \\ &= c_2^2 c_1 s_1 (c_3^2 + s_3^2) - s_2^2 c_1 s_1 (s_3^2 + c_3^2) \\ &= c_2^2 c_1 s_1 - s_2^2 c_1 s_1 \\ &= c_1 s_1 (c_2^2 - s_2^2) \end{aligned}$$

Using (22),

$$p_0 p_1 - e\, p_2 p_3 = \frac{1}{2}\sin\theta_1 \cos\theta_2$$

$$\sin\theta_1 \cos\theta_2 = 2(p_0 p_1 - e\, p_2 p_3) \tag{31}$$

Combining with (30),

$$\tan\theta_1 = \frac{2(p_0 p_1 - e\, p_2 p_3)}{1-2(p_1^2 + p_2^2)} \tag{32}$$

As before, there is a singularity where $\theta_2 = \pm 90°$.

## Singularity at θ₂ = +90°

At $\theta_2 = +90°$,

$$s_2 \equiv \sin\frac{\theta_2}{2} = \sin 45° = \frac{1}{\sqrt{2}}$$
$$c_2 \equiv \cos\frac{\theta_2}{2} = \cos 45° = \frac{1}{\sqrt{2}}$$

From (23),

$$\begin{aligned}
p_0 &= c_3 c_2 c_1 - e\, s_3 s_2 s_1 = \frac{1}{\sqrt{2}}(c_3 c_1 - e\, s_3 s_1) = \frac{1}{\sqrt{2}}(\cos\frac{\theta_3}{2}\cos\frac{\theta_1}{2} - e\sin\frac{\theta_3}{2}\sin\frac{\theta_1}{2})\\
p_1 &= c_3 c_2 s_1 + e\, s_3 s_2 c_1 = \frac{1}{\sqrt{2}}(c_3 s_1 + e\, s_3 c_1) = \frac{1}{\sqrt{2}}(\cos\frac{\theta_3}{2}\sin\frac{\theta_1}{2} + e\sin\frac{\theta_3}{2}\cos\frac{\theta_1}{2})\\
p_2 &= c_3 s_2 c_1 - e\, s_3 c_2 s_1 = \frac{1}{\sqrt{2}}(c_3 c_1 - e\, s_3 s_1) = \frac{1}{\sqrt{2}}(\cos\frac{\theta_3}{2}\cos\frac{\theta_1}{2} - e\sin\frac{\theta_3}{2}\sin\frac{\theta_1}{2})\\
p_3 &= s_3 c_2 c_1 + e\, c_3 s_2 s_1 = \frac{1}{\sqrt{2}}(s_3 c_1 + e\, c_3 s_1) = \frac{1}{\sqrt{2}}(\sin\frac{\theta_3}{2}\cos\frac{\theta_1}{2} + e\cos\frac{\theta_3}{2}\sin\frac{\theta_1}{2})
\end{aligned} \tag{33}$$

$$\begin{aligned}
p_0 &= p_2 = \frac{1}{\sqrt{2}}(\cos\frac{\theta_3}{2}\cos\frac{\theta_1}{2} - e\sin\frac{\theta_3}{2}\sin\frac{\theta_1}{2})\\
p_1 &= e\, p_3 = \frac{1}{\sqrt{2}}(\cos\frac{\theta_3}{2}\sin\frac{\theta_1}{2} + e\sin\frac{\theta_3}{2}\cos\frac{\theta_1}{2})
\end{aligned} \tag{34}$$

If $e = 1$, applying the angle addition formulas gives

$$\begin{aligned}
p_0 &= p_2 = \frac{1}{\sqrt{2}}(\cos\frac{\theta_3}{2}\cos\frac{\theta_1}{2} - \sin\frac{\theta_3}{2}\sin\frac{\theta_1}{2}) = \frac{1}{\sqrt{2}}\cos(\theta_3+\theta_1)\\
p_1 &= p_3 = \frac{1}{\sqrt{2}}(\cos\frac{\theta_3}{2}\sin\frac{\theta_1}{2} + \sin\frac{\theta_3}{2}\cos\frac{\theta_1}{2}) = \frac{1}{\sqrt{2}}\sin(\theta_3+\theta_1)
\end{aligned}$$

Dividing the second equation by the first,

$$\tan(\theta_3+\theta_1) = \frac{p_1}{p_0}$$

We can arbitrarily choose one of the angles. It's convenient to set one of them to zero, so

$$\theta_3 = 0 \qquad \tan(\theta_1) = \frac{p_1}{p_0} \tag{35}$$

If $e = -1$, applying the angle addition formulas to (34) gives

$$\begin{aligned}
p_0 &= p_2 = \frac{1}{\sqrt{2}}(\cos\frac{\theta_3}{2}\cos\frac{\theta_1}{2} + \sin\frac{\theta_3}{2}\sin\frac{\theta_1}{2}) = \frac{1}{\sqrt{2}}\cos(\theta_3-\theta_1) = \frac{1}{\sqrt{2}}\cos(\theta_1-\theta_3)\\
p_1 &= -p_3 = \frac{1}{\sqrt{2}}(\cos\frac{\theta_3}{2}\sin\frac{\theta_1}{2} - \sin\frac{\theta_3}{2}\cos\frac{\theta_1}{2}) = -\frac{1}{\sqrt{2}}\sin(\theta_3-\theta_1) = \frac{1}{\sqrt{2}}\sin(\theta_1-\theta_3)
\end{aligned}$$

Dividing the second equation by the first,

$$\tan(\theta_1 - \theta_3) = \frac{p_1}{p_0}$$

Again, we can arbitrarily choose one of the angles.

$$\theta_3 = 0 \qquad \tan(\theta_1) = \frac{p_1}{p_0} \tag{36}$$

This is the same result as (35), so it holds whether *e* is positive or negative.

## Singularity at $\theta_2$ = -90°

At $\theta_2$ = -90°,

$$\begin{aligned} s_2 &\equiv \sin\frac{\theta_2}{2} = \sin(-45^\circ) = -\frac{1}{\sqrt{2}} \\ c_2 &\equiv \cos\frac{\theta_2}{2} = \cos(-45^\circ) = \frac{1}{\sqrt{2}} \end{aligned}$$

From (23),

$$\begin{aligned} p_0 &= c_3 c_2 c_1 - e s_3 s_2 s_1 = \frac{1}{\sqrt{2}}(c_3 c_1 + e s_3 s_1) = \frac{1}{\sqrt{2}}(\cos\frac{\theta_3}{2}\cos\frac{\theta_1}{2} + e\sin\frac{\theta_3}{2}\sin\frac{\theta_1}{2}) \\ p_1 &= c_3 c_2 s_1 + e s_3 s_2 c_1 = \frac{1}{\sqrt{2}}(c_3 s_1 - e s_3 c_1) = \frac{1}{\sqrt{2}}(\cos\frac{\theta_3}{2}\sin\frac{\theta_1}{2} - e\sin\frac{\theta_3}{2}\cos\frac{\theta_1}{2}) \\ p_2 &= c_3 s_2 c_1 - e s_3 c_2 s_1 = \frac{1}{\sqrt{2}}(-c_3 c_1 - e s_3 s_1) = -\frac{1}{\sqrt{2}}(\cos\frac{\theta_3}{2}\cos\frac{\theta_1}{2} + e\sin\frac{\theta_3}{2}\sin\frac{\theta_1}{2}) \\ p_3 &= s_3 c_2 c_1 + e c_3 s_2 s_1 = \frac{1}{\sqrt{2}}(s_3 c_1 - e c_3 s_1) = \frac{1}{\sqrt{2}}(\sin\frac{\theta_3}{2}\cos\frac{\theta_1}{2} - e\cos\frac{\theta_3}{2}\sin\frac{\theta_1}{2}) \end{aligned} \tag{37}$$

$$\begin{aligned} p_0 &= -p_2 = \frac{1}{\sqrt{2}}(\cos\frac{\theta_3}{2}\cos\frac{\theta_1}{2} - e\sin\frac{\theta_3}{2}\sin\frac{\theta_1}{2}) \\ p_1 &= -e\,p_3 = \frac{1}{\sqrt{2}}(\cos\frac{\theta_3}{2}\sin\frac{\theta_1}{2} + e\sin\frac{\theta_3}{2}\cos\frac{\theta_1}{2}) \end{aligned} \tag{38}$$

The formulas for $p_0$ and $p_1$ are the same as (34) in the previous section, so (36) holds for $\theta_2$ = +90°.

## Summary

From (28), (29), (32) and (36),

$$\begin{aligned} \hat{\boldsymbol{e}}_3 * \hat{\boldsymbol{e}}_2 &\equiv e\,\hat{\boldsymbol{e}}_1 \\ \tan\theta_1 &= \frac{2(p_0 p_1 - e\,p_2 p_3)}{1-2(p_1^2 + p_2^2)} \\ \sin\theta_2 &= 2(p_0 p_2 + e\,p_1 p_3) \\ \tan\theta_3 &= \frac{2(p_0 p_3 - e\,p_1 p_2)}{1-2(p_2^2 + p_3^2)} \\ & \\ &\textit{Except: If } \theta_2 = \pm 90^\circ\textit{:} \\ \theta_3 &= 0 \\ \tan(\theta_1) &= \frac{p_1}{p_0} \end{aligned} \tag{39}$$

**Note:** Although $p_0$ is the scalar component of the quaternion, and $p_1$, $p_2$, $p_3$ are the remaining components, the subscripts indicate the order of rotation. So for example, if the rotation is around the y-axis, then around the x-axis, then around the z-axis, the quaternion is ( $p_0$, $p_2$, $p_1$, $p_3$), not ($p_0$,

$p_1$, $p_2$, $p_3$)!

## Detailed Proof of (18)

The most straightforward way to prove this is to work out every possible combination of our three mutually perpendicular unit vectors.

| *order* | $\hat{e}_3*\hat{e}_2 \equiv e\hat{e}_1$ | $e$ | $\hat{e}_2*\hat{e}_1 = e\hat{e}_1$ | $\hat{e}_3*\hat{e}_1 = -e\hat{e}_1$ | $\hat{e}_3*\hat{e}_2*\hat{e}_1 = -e$ |
|---|---|---|---|---|---|
| *XYZ* | $\hat{k}*\hat{j} \equiv -\hat{i}$ | $-1$ | $\hat{j}*\hat{i} \equiv -\hat{k}$ | $\hat{k}*\hat{i} \equiv +\hat{j}$ | $\hat{k}*\hat{j}*\hat{i} \equiv +1$ |
| *YZX* | $\hat{i}*\hat{k} \equiv -\hat{j}$ | $-1$ | $\hat{k}*\hat{j} \equiv -\hat{i}$ | $\hat{i}*\hat{j} \equiv +\hat{k}$ | $\hat{i}*\hat{k}*\hat{j} \equiv +1$ |
| *ZXY* | $\hat{j}*\hat{i} \equiv -\hat{k}$ | $-1$ | $\hat{i}*\hat{k} \equiv -\hat{j}$ | $\hat{j}*\hat{k} \equiv +\hat{i}$ | $\hat{j}*\hat{i}*\hat{k} \equiv +1$ |
| *ZYX* | $\hat{i}*\hat{j} \equiv +\hat{k}$ | $+1$ | $\hat{j}*\hat{k} \equiv +\hat{i}$ | $\hat{i}*\hat{k} \equiv -\hat{j}$ | $\hat{i}*\hat{j}*\hat{k} \equiv -1$ |
| *XZY* | $\hat{j}*\hat{k} \equiv +\hat{i}$ | $+1$ | $\hat{k}*\hat{i} \equiv +\hat{j}$ | $\hat{j}*\hat{i} \equiv -\hat{k}$ | $\hat{j}*\hat{k}*\hat{i} \equiv -1$ |
| *YXZ* | $\hat{k}*\hat{i} \equiv +\hat{j}$ | $+1$ | $\hat{i}*\hat{j} \equiv +\hat{k}$ | $\hat{k}*\hat{j} \equiv -\hat{i}$ | $\hat{k}*\hat{i}*\hat{j} \equiv -1$ |

(40)

In each case, (18) holds.