Derivation of angular velocity components (pqr) in body coordinates. This derivation applies to Euler angles relative to the right handed coordinate system defined by a set of axes fixed to the Earth. Yaw is the first rotation about the z axis, pitch is the second rotation about the y axis that formed following yaw, and roll is the third rotation about the x axis that was formed following pitch. Yaw is the first rotation about the Earth’s \( \hat{E}_Z \) Axis. This gives, \( \hat{X}_1 \), \( \hat{Y}_1 \), and \( \hat{Z}_1 \) body axes in Earth coordinates:

\[
\begin{bmatrix}
\hat{X}_1 \\
\hat{Y}_1 \\
\hat{Z}_1
\end{bmatrix} = A \cdot 
\begin{bmatrix}
\hat{X}_E \\
\hat{Y}_E \\
\hat{Z}_E
\end{bmatrix} = 
\begin{bmatrix}
\cos \psi & \sin \psi & 0 \\
-\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{bmatrix} 
\begin{bmatrix}
\hat{X}_E \\
\hat{Y}_E \\
\hat{Z}_E
\end{bmatrix}
\]

The \( |A| \) matrix is the direction cosine matrix and defines the axes components after the yaw rotation. The \( |B| \) and \( |C| \) matrices will define the axes components after the pitch and roll rotations. The Earth axes in \( \hat{X}_1 \), \( \hat{Y}_1 \), and \( \hat{Z}_1 \) body coordinates are given by transposing the A matrix:

\[
\begin{bmatrix}
\hat{X}_E \\
\hat{Y}_E \\
\hat{Z}_E
\end{bmatrix} = A^T \cdot 
\begin{bmatrix}
\hat{X}_1 \\
\hat{Y}_1 \\
\hat{Z}_1
\end{bmatrix} = 
\begin{bmatrix}
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{bmatrix} 
\begin{bmatrix}
\hat{X}_1 \\
\hat{Y}_1 \\
\hat{Z}_1
\end{bmatrix}
\]

This says that:

\[
\begin{align*}
\hat{X}_E &= \hat{X}_1 \cos \theta - \hat{Y}_1 \sin \theta \\
\hat{Y}_E &= \hat{X}_1 \sin \theta + \hat{Y}_1 \cos \theta \\
\hat{Z}_E &= \hat{Z}_1
\end{align*}
\]

Pitch is defined as the rotation about the new \( \hat{Y}_1 \) axis that was created when we yawed about the \( \hat{Z}_E \) axis. We pitch about the \( \hat{Y}_1 \) axis to get \( \hat{X}_2 \), \( \hat{Y}_2 \), \( \hat{Z}_2 \) body axes.

\[
\begin{bmatrix}
\hat{X}_2 \\
\hat{Y}_2 \\
\hat{Z}_2
\end{bmatrix} = B \cdot 
\begin{bmatrix}
\hat{X}_1 \\
\hat{Y}_1 \\
\hat{Z}_1
\end{bmatrix} = 
\begin{bmatrix}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{bmatrix} 
\begin{bmatrix}
\hat{X}_1 \\
\hat{Y}_1 \\
\hat{Z}_1
\end{bmatrix}
\]

Body \( \hat{X}_1 \), \( \hat{Y}_1 \), \( \hat{Z}_1 \) axes in \( \hat{X}_2 \), \( \hat{Y}_2 \), \( \hat{Z}_2 \) coordinates are given by:

\[
\begin{bmatrix}
\hat{X}_1 \\
\hat{Y}_1 \\
\hat{Z}_1
\end{bmatrix} = B^T \cdot 
\begin{bmatrix}
\hat{X}_2 \\
\hat{Y}_2 \\
\hat{Z}_2
\end{bmatrix} = 
\begin{bmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{bmatrix} 
\begin{bmatrix}
\hat{X}_2 \\
\hat{Y}_2 \\
\hat{Z}_2
\end{bmatrix}
\]

This says that:

\[
\begin{align*}
\hat{X}_1 &= \hat{X}_2 \cos \theta + \hat{Z}_2 \sin \theta \\
\hat{Y}_1 &= \hat{Y}_2 \\
\hat{Z}_1 &= -\hat{X}_2 \sin \theta + \hat{Z}_2 \cos \theta
\end{align*}
\]
Roll is defined as the rotation about the new \( \hat{X}_2 \) axis that was created when we pitched. We roll about the \( \hat{X}_2 \) axis to give us the final \( \hat{X}_T, \hat{Y}_T, \hat{Z}_T \) body axes in Earth coordinates.

\[
\begin{bmatrix}
\dot{X}_T \\
\dot{Y}_T \\
\dot{Z}_T
\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix}
\hat{X}_2 \\
\hat{Y}_2 \\
\hat{Z}_2
\end{bmatrix}
\]

\( \dot{X}_T = \begin{bmatrix} \cos \phi & \sin \phi & 0 \end{bmatrix} \begin{bmatrix} \hat{X}_2 \\
\hat{Y}_2 \\
\hat{Z}_2
\end{bmatrix} \)  

\( \dot{Y}_T = -\sin \phi \cos \phi \) \( \dot{Z}_T = \begin{bmatrix} \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} \hat{X}_2 \\
\hat{Y}_2 \\
\hat{Z}_2
\end{bmatrix} \)  

Equation 7

Body \( \hat{X}_2, \hat{Y}_2, \hat{Z}_2 \) axes in \( \hat{X}_T, \hat{Y}_T, \hat{Z}_T \) coordinates are given by:

\[
\begin{bmatrix}
\hat{X}_2 \\
\hat{Y}_2 \\
\hat{Z}_2
\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix}
\hat{X}_T \\
\hat{Y}_T \\
\hat{Z}_T
\end{bmatrix} 
\]

\( \hat{X}_2 = \hat{X}_T \)  

\( \hat{Y}_2 = \hat{Y}_T \cos \phi - \hat{Z}_T \sin \phi \)  

\( \hat{Z}_2 = \hat{Y}_T \sin \phi + \hat{Z}_T \cos \phi \)  

Equation 8

This says that:

\[
\begin{align*}
\dot{X}_2 &= \hat{X}_T \\
\dot{Y}_2 &= \hat{Y}_T \cos \phi - \hat{Z}_T \sin \phi \\
\dot{Z}_2 &= \hat{Y}_T \sin \phi + \hat{Z}_T \cos \phi
\end{align*}
\]

Equation 9

And

\[
\begin{align*}
\dot{X}_T &= \begin{bmatrix} X_E \end{bmatrix} \\
\dot{Y}_T &= \begin{bmatrix} Y_E \end{bmatrix} \\
\dot{Z}_T &= \begin{bmatrix} Z_E \end{bmatrix}
\end{align*}
\]

Equation 10

Each rotation can be considered independently about the \( \hat{Z}_E, \hat{Y}_1, \hat{X}_2 \) axes respectively. Therefore, their rates can be summed to derive an angular velocity vector.

\[
\omega = \dot{\psi}\hat{E}_E + \dot{\theta}\hat{Y}_1 + \dot{\phi}\hat{X}_2
\]

Equation 11

Where \( \dot{\psi} \) is the yaw rate, \( \dot{\theta} \) is the pitch rate, and \( \dot{\phi} \) is the roll rate.

To create a velocity vector in body coordinates, we need to express the angular velocity components in Equation 11 in body coordinates \( \hat{X}_T, \hat{Y}_T, \hat{Z}_T \) to create the angular velocity (pqr) vector. Consider Equations 3 and 6:

\[
\begin{align*}
\dot{Z}_E &= \dot{Z}_i = -\hat{X}_2 \sin \theta + \hat{Z}_2 \cos \theta \\
\dot{Y}_E &= \dot{Y}_i = \hat{X}_2 \sin \theta(\hat{Y}_T \sin \phi + \hat{Z}_T \cos \phi) \cos \theta \\
\dot{X}_E &= \dot{X}_i = -\hat{X}_2 \sin \theta + \hat{Y}_2 \sin \phi \cos \theta + \hat{Z}_2 \cos \phi \cos \theta
\end{align*}
\]

Equation 12

Substitute \( \dot{X}_2 \) and \( \dot{Z}_2 \) from Equations 9, you get

\[
\begin{align*}
\dot{Z}_E &= \dot{Z}_i = -\hat{X}_2 \sin \theta + (\hat{Y}_r \sin \phi + \hat{Z}_r \cos \phi) \cos \theta \\
\dot{Y}_E &= \dot{Y}_i = \hat{Z}_2 \sin \phi \cos \theta + \hat{Y}_2 \sin \phi \cos \theta + \hat{Z}_2 \cos \phi \cos \theta
\end{align*}
\]

Equation 13

Now, we have \( \dot{Z}_E \) in terms of body coordinates. Now consider the second component of Equation 11. Substitute the \( \dot{Y}_1 \) using Equations 6 and 9:

\[
\dot{Y}_1 = \dot{Y}_2 = \hat{Y}_T \cos \phi - \hat{Z}_T \sin \phi
\]

Equation 14
Now, we have $\hat{Y}_1$ in terms of body coordinates. And finally consider the third component of Equation 11. Substitute the $\hat{X}_2$ with $\hat{X}_T$ from Equation 9:

$$\hat{X}_2 = \hat{X}_T$$

Substitute the unit vectors in equation 10 with equations 13, 14, and 15.

$$\omega = \dot{\psi}(-\hat{X}_T \sin \theta + \hat{Y}_T \sin \phi \cos \theta + \hat{Z}_T \cos \phi \cos \theta) + \dot{\theta}(\dot{\hat{Y}}_T \cos \phi - \dot{\hat{Z}}_T \sin \phi) + \dot{\phi}(\dot{\hat{X}}_T)$$

Now rearrange Equation 16 to combine the $\hat{X}_T$, $\hat{Y}_T$, $\hat{Z}_T$ components so

$$\omega = \hat{X}_T (\dot{\phi} - \dot{\psi} \sin \theta) + \hat{Y}_T (\dot{\theta} \cos \phi + \dot{\psi} \sin \phi \cos \theta) + \hat{Z}_T (-\dot{\theta} \sin \phi + \dot{\psi} \cos \phi \cos \theta)$$

Angular velocity is a vector with components that are functions of pitch, roll, yaw rate, pitch rate, and roll rate. The resulting components agree with the $\omega$, $\omega$, and $\omega$ values given in the IEEE std. P1278.1/d8.