Quaternion differentiation

Quaternion differentiation’s formula connects time derivative of component of quaternion \( q(t) \) with component of vector of angular velocity \( W(t) \). Quaternion \( q(t) = (q_0(t), q_1(t), q_2(t), q_3(t)) \) determines attitude of rigid body moving with one fixed point, vector of angular velocity \( W(t) = (W_x(t), W_y(t), W_z(t)) \) determines angular velocity of this body at point of time \( t \). Norm of quaternion \( q(t) \) is unit, i.e.

\[ q_0^2(t) + q_1^2(t) + q_2^2(t) + q_3^2(t) = 1 \]

(1)

Vector \( W(t) \) can be represented as quaternion with zero scalar part, i.e.

\[ W(t) = (0, W_x(t), W_y(t), W_z(t)) \]

(2)

Quaternion differentiation’s formula can be represented as

\[ \frac{dq(t)}{dt} = \frac{1}{2} W(t) q(t) \]

(3)

Using quaternion multiplication rule

\[ \frac{dq_0(t)}{dt} = -\frac{1}{2} (W_x(t)q_1(t) + W_y(t)q_2(t) + W_z(t)q_3(t)) \]

\[ \frac{dq_1(t)}{dt} = \frac{1}{2} (W_x(t)q_0(t) + W_y(t)q_3(t) - W_z(t)q_2(t)) \]

\[ \frac{dq_2(t)}{dt} = \frac{1}{2} (W_y(t)q_0(t) + W_z(t)q_1(t) - W_x(t)q_3(t)) \]

\[ \frac{dq_3(t)}{dt} = \frac{1}{2} (W_z(t)q_0(t) + W_x(t)q_2(t) - W_y(t)q_1(t)) \]

(4)

Below formula (3) brief and vigorous derivation follows.

Let \( R_0 \) is any given vector (quaternion with zero scalar part) fixed in rigid body at initial moment of time \( t_0 \), \( R_t \) is the same vector (quaternion) at moment of time \( t \). Then, obviously

\[ R_t = q(t) R_0 q^{-1}(t) \]

(5)

If we differentiate (5) then

\[ \frac{dR_t}{dt} = \frac{dq(t)}{dt} R_0 q^{-1}(t) + q(t) R_0 \frac{dq^{-1}(t)}{dt} \]
From (5) and (6) we have

\[
\frac{dR_t}{dt} = \frac{dq(t)}{dt} q^{-1}(t) R_t + R_t \frac{dq^{-1}(t)}{dt}
\]

(7)

Because norm of quaternion \(q(t)\) is unit, i.e.

\[
q(t)q^{-1}(t) = 1
\]

(8)

we have

\[
\frac{dq(t)}{dt} q^{-1}(t) + q(t) \frac{dq^{-1}(t)}{dt} = 0
\]

(9)

It follows from (7) and (9) that

\[
\frac{dR_t}{dt} = \frac{dq(t)}{dt} q^{-1}(t) R_t - R_t \frac{dq(t)}{dt} q^{-1}(t)
\]

(10)

Let

\[
p(t) = \frac{dq(t)}{dt} q^{-1}(t)
\]

(11)

Obviously

\[
q^{-1}(t) = S(q^{-1}(t)) + V(q^{-1}(t)) = q_0(t) - (q_1(t), q_2(t), q_3(t))
\]

(12)

where \(S(\cdot)\) = scalar part, \(V(\cdot)\) = vector part of quaternion

Scalar part \(S(p(t))\) of quaternion \(p(t)\) equals

\[
S(p(t)) = \frac{dq_0(t)}{dt} q_0 + \frac{dq_1(t)}{dt} q_1 + \frac{dq_2(t)}{dt} q_2 + \frac{dq_3(t)}{dt} q_3 = 0
\]

because norm of quaternion \(q(t)\) is unit.

It follows that \(p(t)\) is vector. Because \(R_t\) is also vector

\[
p(t)R_t - R_t p(t) = 2[p(t)R_t]
\]

(13)

where \([ab]\) = cross-product of vector \(a\) and vector \(b\).
On the other hand, variable \( \frac{dR}{dt} \) is velocity of point of rigid body with one fixed point. Hence

\[
\frac{dR}{dt} = [W(t)R]_{v}
\]

(14).

Because \( R_{t} \) is arbitrary vector, it follows from (10), (11), (13) and (14)

\[
W(t) = 2p(t) = 2\frac{dq(t)}{dt}q^{-1}(t)
\]

(15)

It follows from (15) finally that

\[
\frac{dq(t)}{dt} = \frac{1}{2}W(t)q(t)
\]

(3)

It is necessary to draw attention: in formula (3) angular velocity vector \( W(t)=(W_{X}(t), W_{Y}(t), W_{Z}(t)) \) is represented by projections on axes of unmoving system of coordinates. If we apply the projections on axes of moving system of coordinates for the same angular velocity vector then obviously:

\[
(0,W_{X}(t),W_{Y}(t),W_{Z}(t)) = q(t)(0,W_{x}(t),W_{y}(t),W_{z}(t))q^{-1}(t)
\]

(16)

Where \( W_{X}(t), W_{Y}(t), W_{Z}(t) \) are projections of angular velocity vector on axes of moving system of coordinates.

It follows from (3) and (16) that

\[
\frac{dq(t)}{dt} = \frac{1}{2}q(t)\bar{W}(t)
\]

\[
\bar{W}(t) = (W_{x}(t),W_{y}(t),W_{z}(t))
\]

(17)

Finally, consider example of formulas (3) and (17) using for so-called conic moving. In this case quaternion \( q(t) \) equals:

\[
q(t) = (\cos\left(\frac{\beta}{2}\right), \sin\left(\frac{\beta}{2}\right)(\cos(\omega t),\sin(\omega t),0))
\]

\[
q^{-1}(t) = (\cos\left(\frac{\beta}{2}\right),-\sin\left(\frac{\beta}{2}\right)(\cos(\omega t),\sin(\omega t),0))
\]

\[
\frac{dq(t)}{dt} = (0, \omega \sin\left(\frac{\beta}{2}\right)(-\sin(\omega t),\cos(\omega t),0))
\]

(18)
Accordingly (3) and (17) projections of angular velocity vector on axes of unmoving and moving systems of coordinates equals:

\[
(W_x(t), W_y(t), W_z(t)) = 2 \frac{dq(t)}{dt} - q^{-1}(t) = (-\omega \sin(\beta) \sin(\omega t), \omega \sin(\beta) \cos(\omega t), \omega (1 - \cos(\beta)))
\]

\[
(W_x(t), W_y(t), W_z(t)) = 2q^{-1}(t) \frac{dq(t)}{dt} = (-\omega \sin(\beta) \sin(\omega t), \omega \sin(\beta) \cos(\omega t), \omega (\cos(\beta) - 1))
\]