Quaternion differentiation

Quaternion differentiation's formula connects time derivative of component of quaternion q(t) with component of vector of angular velocity W(t). Quaternion q(t)=(q_0(t), q_1(t), q_2(t), q_3(t)) determines attitude of rigid body moving with one fixed point, vector of angular velocity W(t)=(W_x(t), W_y(t), W_z(t)) determines angular velocity of this body at point of time t. point of time. Norm of quaternion q(t) is unit, i.e.

$$q_0^2(t) + q_1^2(t) + q_2^2(t) + q_3^2(t) = 1$$

(1)

Vector W(t) can be represented as quaternion with zero scalar part, i.e

$$W(t) = (0, W_x(t), W_y(t), W_z(t))$$

(2)

Quaternion differentiation's formula can be represented as

$$\frac{dq(t)}{dt} = \frac{1}{2} W(t) q(t)$$

(3)

Using quaternion multiplication rule

$$\frac{dq_0(t)}{dt} = -\frac{1}{2}(W_x(t)q_1(t) + W_y(t)q_2(t) + W_z(t)q_3(t))$$
$$\frac{dq_1(t)}{dt} = \frac{1}{2}(W_x(t)q_0(t) + W_y(t)q_3(t) - W_z(t)q_2(t))$$
$$\frac{dq_2(t)}{dt} = \frac{1}{2}(W_y(t)q_0(t) + W_z(t)q_1(t) - W_x(t)q_3(t))$$
$$\frac{dq_3(t)}{dt} = \frac{1}{2}(W_z(t)q_0(t) + W_x(t)q_2(t) - W_y(t)q_1(t))$$

(4)

Below formula (3) brief and vigorous derivation follows.

Let R_0 is any given vector (quaternion with zero scalar part) fixed in rigid body at initial moment of time t_0, R_t is the same vector (quaternion) at moment of time t. Then, obviously

$$R_t = q(t) R_0 q^{-1}(t)$$

(5)

If we differentiate (5) then

$$\frac{dR_t}{dt} = \frac{dq(t)}{dt} R_0 q^{-1}(t) + q(t) R_0 \frac{dq^{-1}(t)}{dt}$$

(6)

From (5) and (6) we have

$$\frac{dR_t}{dt} = \frac{dq(t)}{dt} q^{-1}(t) R_t + R_t q(t) \frac{dq^{-1}(t)}{dt}$$

(7)

Because norm of quaternion q(t) is unit, i.e.

$$q(t) q^{-1}(t) = 1$$

(8)

we have

$$\frac{dq(t)}{dt} q^{-1}(t) + q(t) \frac{dq^{-1}(t)}{dt} = 0$$

(9)

It follows from (7) and (9) that

$$\frac{dR_t}{dt} = \frac{dq(t)}{dt} q^{-1}(t) R_t - R_t \frac{dq(t)}{dt} q^{-1}(t)$$

(10)

Let

$$p(t) = \frac{dq(t)}{dt} q^{-1}(t)$$

(11)

Obviously

$$q^{-1}(t) = S(q^{-1}(t)) + V(q^{-1}(t)) = q_0(t) - (q_1(t), q_2(t), q_2(t))$$

(12)

where S() =scalar part, V()=vector part of quaternion

Scalar part S(p(t)) of quaternion p(t) equals

$$S(p(t)) = \frac{dq_0(t)}{dt} q_0 + \frac{dq_1(t)}{dt} q_1 + \frac{dq_2(t)}{dt} q_2 + \frac{dq_3(t)}{dt} q_3 = 0$$

because norm of quaternion q(t) is unit.

It follows that p(t) is vector. Because R_t is also vector

$$p(t) R_t - R_t p(t) = 2[p(t) R_t]$$

(13)

where [ab]= cross-product of vector a and vector b.

On the other hand, variable $\frac{dR_t}{dt}$ is velocity of point of rigid body with one fixed point. Hence

$$\frac{dR_t}{dt} = [W(t)R_t]$$

(14).

Because R_t is arbitrary vector, it follows from (10), (11), (13) and (14)

$$W(t) = 2p(t) = 2\frac{dq(t)}{dt}q^{-1}(t)$$

(15)

It follows from (15) finally that

$$\frac{dq(t)}{dt} = \frac{1}{2}W(t)q(t)$$

(3)

It is necessary to draw attention: in formula (3) angular velocity vector $W(t)=(W_X(t), W_Y(t), W_Z(t))$ is represented by projections on axes of unmoving system of coordinates. If we apply the projections on axes of moving system of coordinates for the same angular velocity vector then obviously:

$$(0, W_X(t), W_Y(t), W_Z(t)) = q(t)(0, W_x(t), W_y(t), W_z(t))q^{-1}(t)$$

(16)

Where $W_x(t)$, $W_y(t)$, $W_z(t)$ are projections of angular velocity vector on axes of moving system of coordinates.

It follows from (3) and (16) that

$$\frac{dq(t)}{dt} = \frac{1}{2}q(t)\overline{W}(t)$$

$$\overline{W}(t) = (W_x(t), W_y(t), W_z(t))$$

(17)

Finally, consider example of formulas (3) and (17) using for so-called conic moving. In this case quaternion q(t) equals:

$$q(t) = (\cos\left(\frac{\beta}{2}\right), \sin\left(\frac{\beta}{2}\right)(\cos(\omega t), \sin(\omega t), 0))$$

$$q^{-1}(t) = (\cos\left(\frac{\beta}{2}\right), -\sin\left(\frac{\beta}{2}\right)(\cos(\omega t), \sin(\omega t), 0))$$

$$\frac{dq(t)}{dt} = (0, \omega\sin\left(\frac{\beta}{2}\right)(-\sin(\omega t), \cos(\omega t), 0))$$

(18)

Accordingly (3) and (17) projections of angular velocity vector on axes of unmoving and moving systems of coordinates equals:

$$(W_X(t), W_Y(t), W_Z(t)) = 2\frac{dq(t)}{dt}q^{-1}(t) = (-\omega\sin(\beta)\sin(\omega t), \omega\sin(\beta)\cos(\omega t), \omega(1-\cos(\beta)))$$

$$(W_x(t), W_y(t), W_z(t)) = 2q^{-1}(t)\frac{dq(t)}{dt} = (-\omega\sin(\beta)\sin(\omega t), \omega\sin(\beta)\cos(\omega t), \omega(\cos(\beta)-1))$$