QUATERNIONS:

What do we do with them? Why do We Need Them?

What are they? How do we make them?

What do we do with them (revisited)?

What are the alternatives? Why quaternions?

Quaternions: What do we do with them?

1) Propagate Attitude

- Calculate attitude of Spacecraft from One Moment to the Next
- Integrate the Spacecraft Equations of Motion

2) Perform Coordinate Transformations

- Calculate Vector in Spacecraft Frame from Vector Known in Inertial Frame

3) Perform Vector Rotations

- Calculate Vector in Inertial Frame from Vector Known in Spacecraft Frame

Quaternions

What are they?

Formulation

Physical Interpretation

Mathematical Details

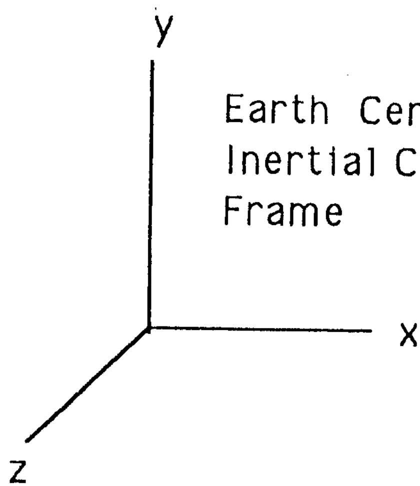
multipication successive rotation
inverse equivalence
two coordinate frames
cross product

Some Operations Using Quaternions

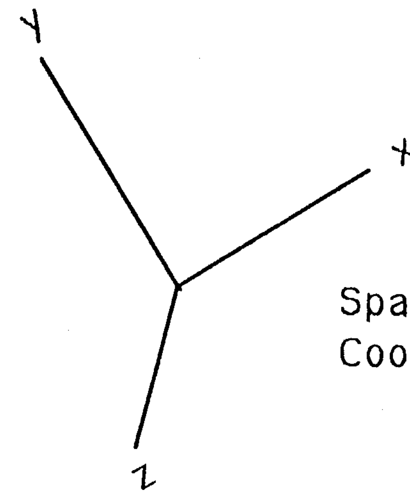
vector rotation, transformation
attitude propagation
spacecraft maneuver
attitude reference update
appendage pointing

Alternatives

Quaternions: What do We do with them?

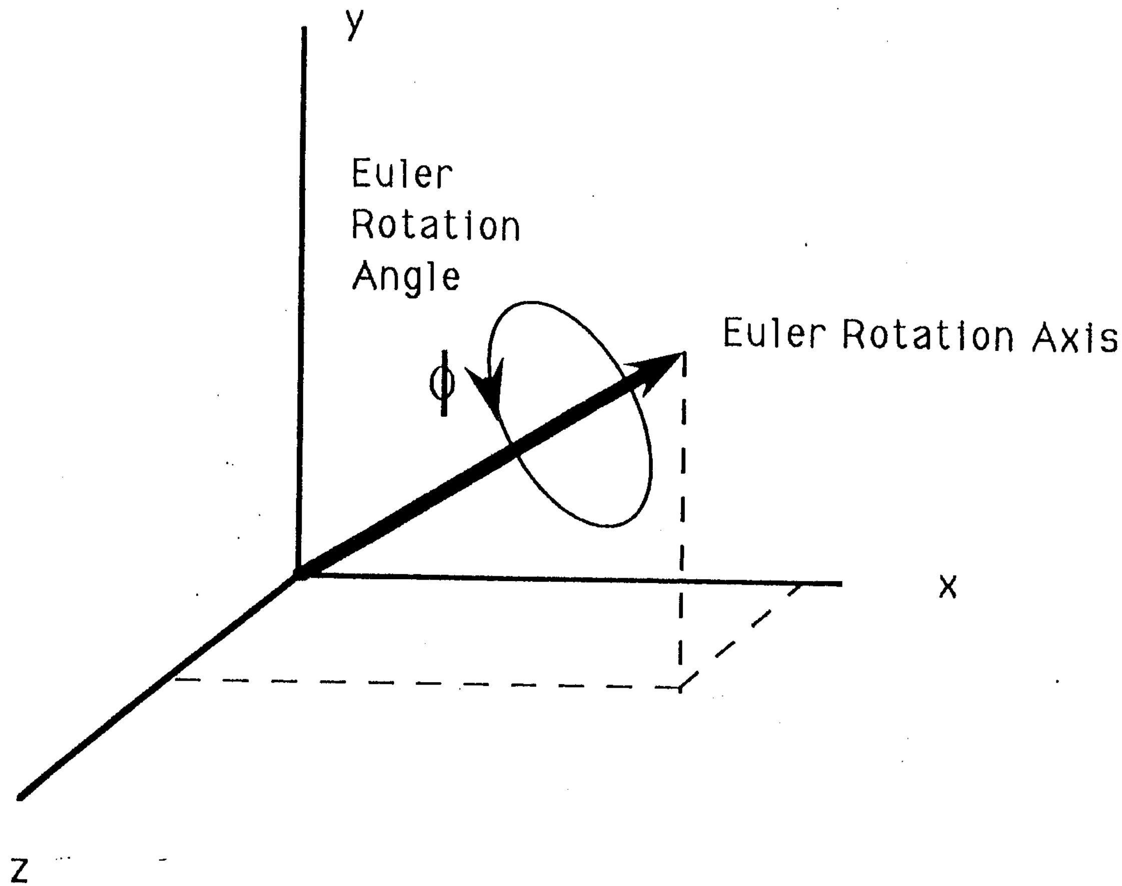


Earth Centered Inertial Coordinate Frame



Spacecraft Fixed Coordinate Frame

Describe the attitude of the Spacecraft, ie, the orientation of the spacecraft coordinate frame relative to an inertially fixed frame.



Quaternion Rotation
Concept

0, 0, 0, 1

QUATERNIONS: What are they? How do we make them?

Euler's Theorem:

ANY FINITE ROTATION OF A RIGID BODY CAN BE EXPRESSED AS A ROTATION
THROUGH SOME ANGLE ABOUT A FIXED AXIS

Rotation Axis unit vector: $e_1,\ e_2,\ e_3$

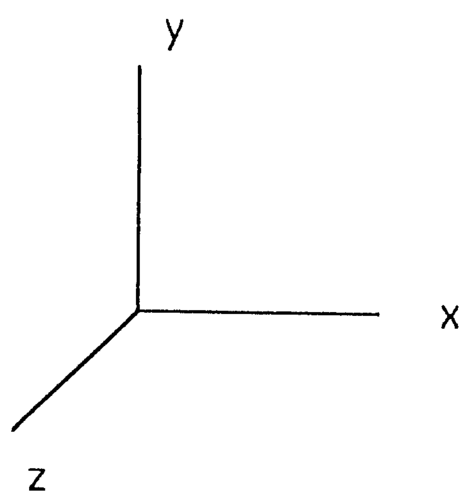
Rotation Angle: ϕ

Quaternion: $e_1 \sin\frac{\phi}{2},\ e_2 \sin\frac{\phi}{2},\ e_3 \sin\frac{\phi}{2},\ \cos\frac{\phi}{2}$

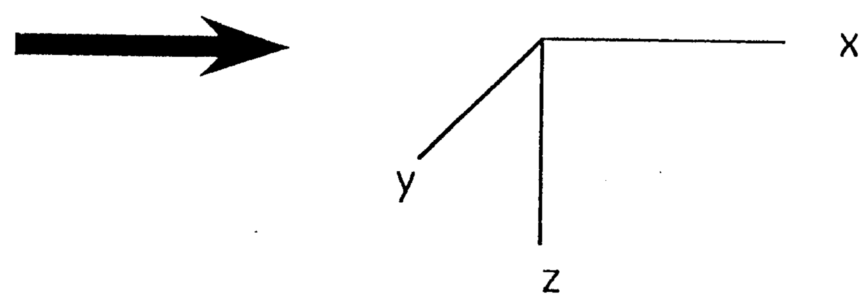
or

$$q_1 i,\ q_2 j,\ q_3 k,\ q_4$$

Example: 90 degree rotation about X axis



rotation axis = 1, 0, 0
rotation angle = 90 degrees



Q = 1*sin(45), 0*sin(45), 0*sin(45), cos(45)

= .7071, 0, 0, .7071

QUATERNION PHYSICAL INTERPRETATION

FOURTH ELEMENT:

COS OF 1/2 ROTATION ANGLE - $\phi/2$

$\phi/2 \sim 1$ => SMALL ANGLE

$\phi/2 \sim 0$ => $\phi/2$ APPROACHING 90 DEG

=> ϕ APPROACHING 180

$\phi/2 < 0$ => ϕ >180

FIRST THREE ELEMENTS:

EIGEN AXIS X SIN OF 1/2 ROTATION ANGLE

RELATIVE AMOUNT OF ROLL, PITCH, YAW OF THE ROTATION

VERY USEFUL FOR RELATING ONE FRAME (ATTITUDE, POSITION, ETC.) TO ANOTHER

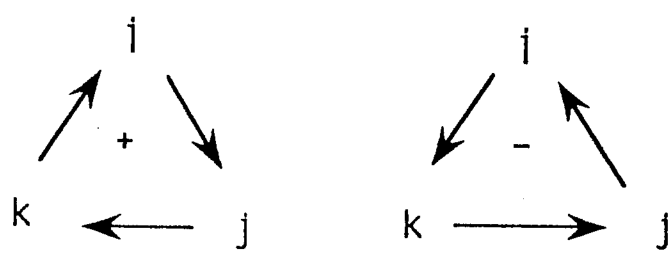
LESS USEFUL FOR ABSOLUTE ATTITUDE INTERPRETATION (NOBODY REMEMBERS WHERE ECI IS)

Quaternion Multiplication

$$(q_1 i,\ q_2 j,\ q_3 k,\ q_4) * (q_1 i,\ q_2 j,\ q_3 k,\ q_4)$$

Multiply term by term

Rules:



$$i^2 = -1$$

$$j^2 = -1$$

$$k^2 = -1$$

Successive Rotations

Successive rotations performed by post multiplying

Given: Rotations $Q_1, Q_2, Q_3, \ldots, Q_n$

$$Q_{1-n} = Q_1 Q_2 Q_3 \ldots Q_n$$

Q_{1-n} is the single quaternion equivalent to all n rotations

Quaternion Inverse

$Q^{-1} = Q^*$ called "conjugate" -
reverse the signs of either the first three or last element

if $Q = q_1, q_2, q_3, q_4$ then $Q^* = q_1, q_2, q_3, -q_4 = -q_1, -q_2, -q_3, q_4$

$$Q^*Q = QQ^* = [0, 0, 0, 1]$$

Conceptually, Q^* represents a rotation opposite to that represented by Q

Equivalent Quaternions

Reversing signs of all four elements of a quaternion yields
An Equivalent Quaternion

$$-Q = Q$$

One Frame Relative to Another

Given two coordinate frames defined by Q_1 and Q_2

$$Q_2 = Q_1 Q_{12}$$

$$Q^*_1 Q_2 = Q^*_1 Q_1 Q_{12}$$

or

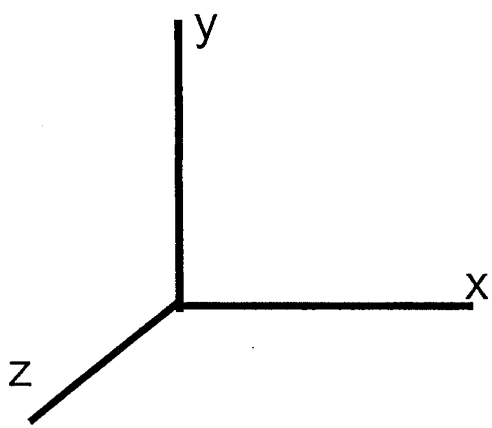
$$Q_{12} = Q^*_1 Q_2$$

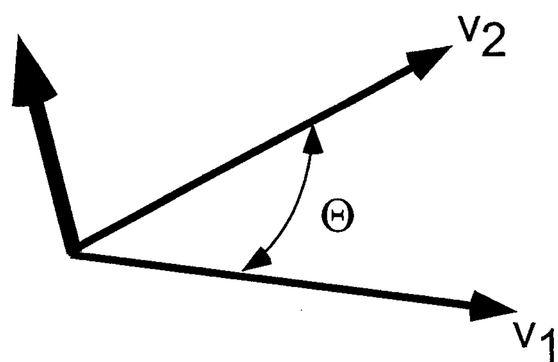
PI

Vector Cross Product and Quaternions

We have two vectors,
we can form a quaternion
that rotates the frame to
make the vectors congruent

$C = v_1 \mathbf{X} v_2$ - magnitude = $|v_1| \; |v_2| \sin\Theta$





if v_1, v_2 are normal vectors
$|C| = \sin\Theta$

$$Q = \hat{C} \sin\Theta/2 \, , \cos\Theta/2$$

$\hat{C}$ = normalized C

<u>ALWAYS Normalize Quaternions</u>

Vector Rotation

We know a vector in the Spacecraft Frame,
Where is it pointing in inertial Frame?

$$V_I = Q^* V_s Q$$

Where:

- V_s = Vector in S/C Frame
- V_I = Vector in Inertial Frame
- Q = S/C Attitude Quaternion
- Q^* = Quaternion Conjugate (Switch signs of first 3 components)

eg: Given S/C Attitude and S/C Frame Star Tracker Vector,
Where is the Star Tracker Pointed in Inertial Frame?

Coordinate Transformation

We Know a Vector in the Inertial Frame,
What is it in the Spacecraft Frame?

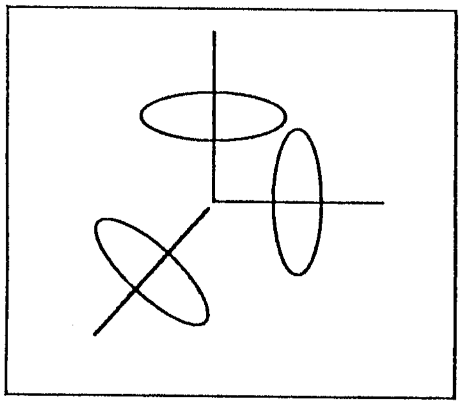
$$V_s = Q\, V_I\, Q^*$$

Where:

V_s = Vector in S/C Frame
V_I = Vector in Inertial Frame
Q = S/C Attitude Quaternion
Q^* = Quaternion Conjugate (Switch signs of first 3 components)

eg: Given S/C Attitude, Earth and S/C Ephemeris,
Where do we Point the Payload relative to the S/C?

Propagate Attitude



Spacecraft
Gyro Package

Spacecraft
Rotational Rates

$\boldsymbol{\omega}$ =
$W_1,\ W_2,\ W_3,\ 0$

Quaternion
Integration

Vehicle
Attitude

$\boldsymbol{\omega}$ = S/C Rotation Rate Vector with zero appended, to make a 4 element entity compatible w/ quaternion multiplications

Q = S/C Attitude Quaternion

$$\dot{Q} = 1/2\, Q\, \boldsymbol{\omega}$$

$\dot{Q}$ = S/C Attitude Quaternion Derivative

Propagating Attitude

Propagate Attitude Using Body Rate Vector ω

<u>in simulation</u>

$$\dot{\omega} = I^{-1} (\mathbf{M} - \omega \times I\omega)$$

$$\omega = \int \dot{\omega}$$

I = 3X3 inertia matrix
M = 3 component moment vector
ω = body rotation rate vector

<u>aboard vehicle on orbit</u>

ω from on board sensors, typically gyros

$$\dot{Q} = 1/2\, Q\, \Omega$$

zero fourth element added

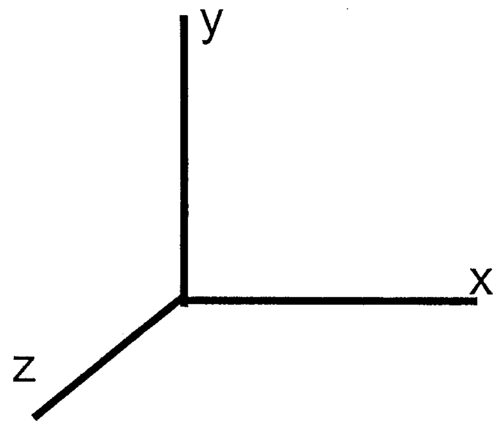
to ω to form "quaternion" Ω

$\dot{Q}$ is integrated to calculate attitude at any moment

Spacecraft Maneuver

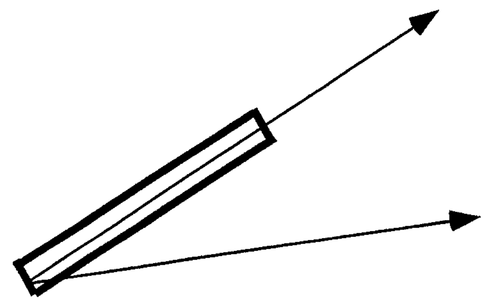
Need to maneuver spacecraft
to point vector (sensor?) at target.

Spacecraft Coordinate Frame



commanded attitude = $Q_c = Q_r$

v_b sensor boresight



v_t vector to target based on Q_r, ephemeris, etc.

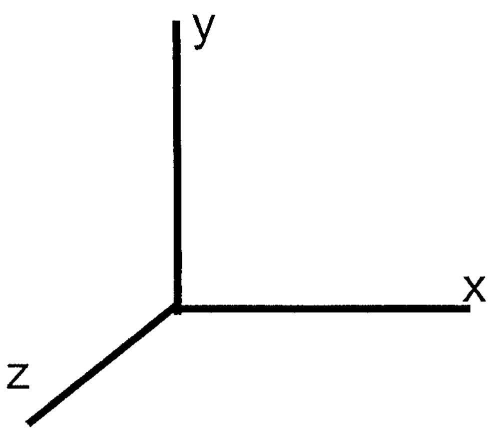
form delta quaternion Q_d by crossing v_b into v_t

new $Q_c' = Q_c \, Q_d$

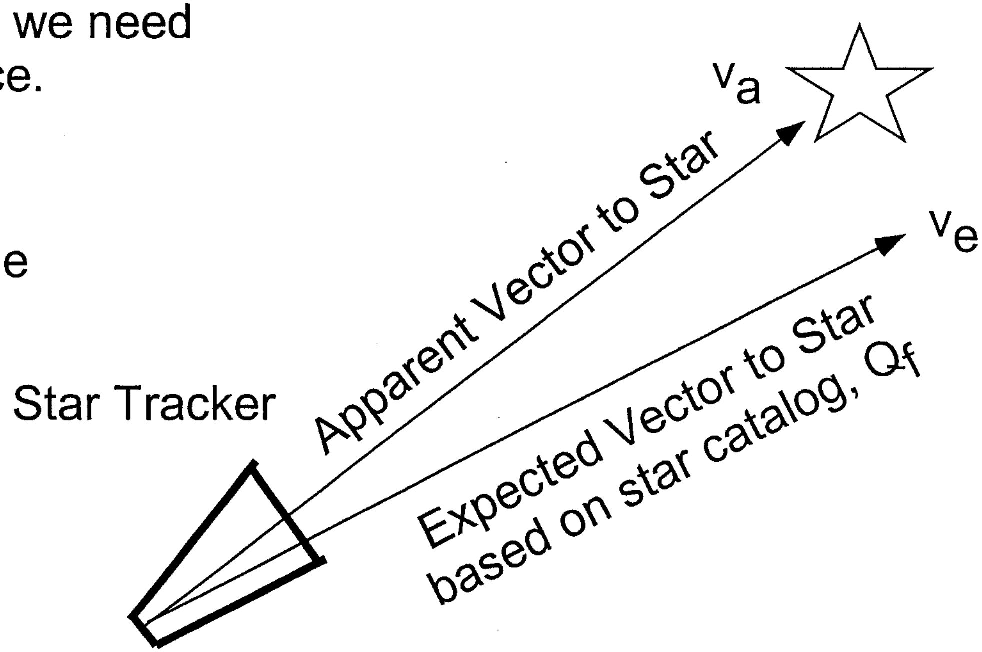
Attitude Reference Update

We have vector from sensor (star tracker, three axis magnetometer, etc) we need to correct our attitude reference.

Spacecraft Coordinate Frame



Attitude = Q_r



form delta quaternion Q_d from v_e crossed into v_a

corrected $Q_r' = Q_r Q_d$

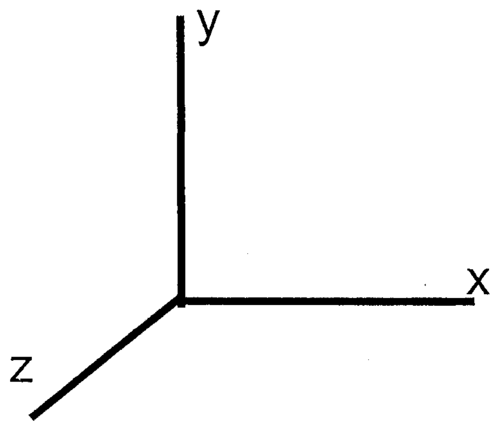
If commanded attitude is not changed, spacecraft will rotate.
If commanded attitude is changed by Q_d^* spacecraft will remain fixed.

Gimbaled Appendage

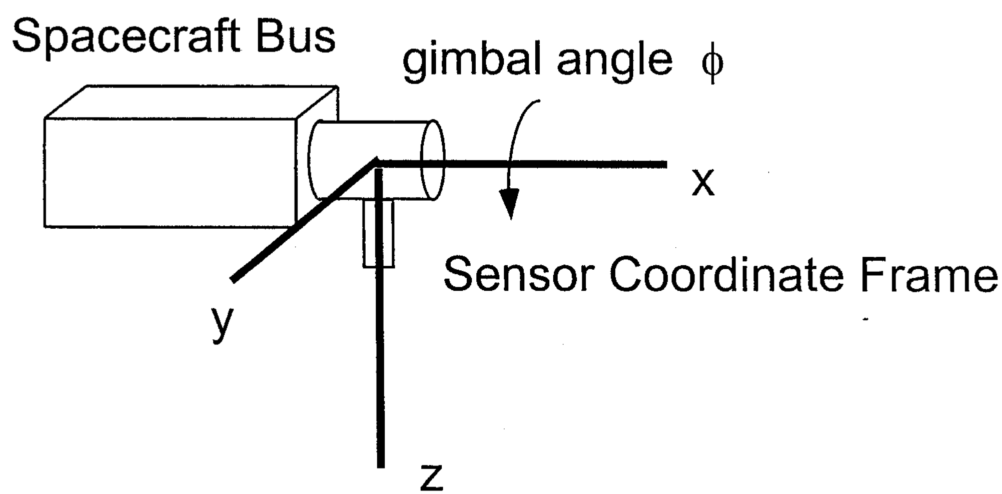
Single Axis Gimbal

Common for LEO Earth observation spacecraft, attitude maintained with x axis along orbit track. Sensor gimbaled to sweep a swath parallel to ground track.

Spacecraft Coordinate Frame



Attitude = Q_r



Spacecraft to sensor $Q_{ss} = [1,0,0]\sin(90+\phi), \cos\phi$

Sensor coordinate frame (relative to ECI)

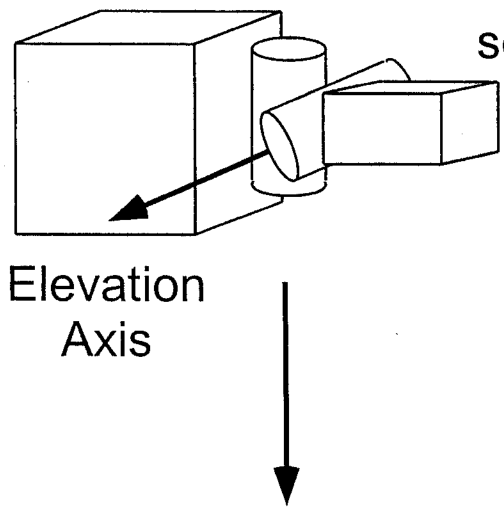
$$Q_s = Q_r Q_{ss}$$

Gimbaled Appendage

Dual Axis Gimbal

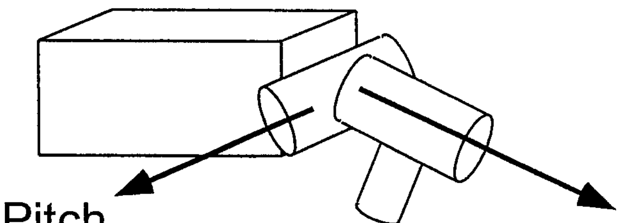
One axis always "carried" on the other

Azimuth-Elevation
Gimbal



Elevation Axis
Carried on Azimuth

Pitch-Roll
Gimbal

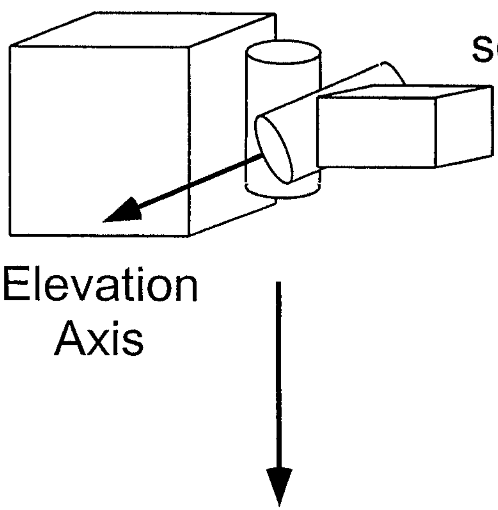


Roll Axis
Carried on Pitch

Gimbaled Appendage

Dual Axis Gimbal

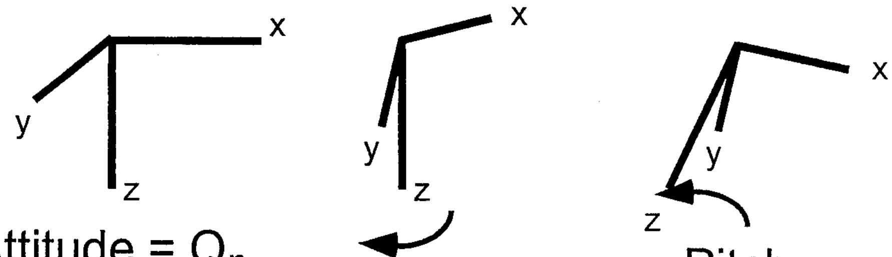
Azimuth-Elevation Gimbal



Spacecraft Coordinate Frame

Intermediate Coordinate Frame

Sensor Coordinate Frame



Attitude = Q_r

Aximuth Angle ϕ

Pitch Angle θ

Elevation Axis Carried on Azimuth

Q_{si} = [0,0,1]sinϕ/2 , cosϕ/2 spacecraft to intermediate

Q_{is} = [0,1,0]sinθ/2 , cosθ/2 intermediate to sensor

$Q_{ss} = Q_{si}Q_{is}$ spacecraft to sensor

$Q_s = Q_rQ_{ss}$ ECI to sensor